A NOTE ON A PAPER BY S. HABER

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ABSTRACT. A technique used by S. Haber to prove an elementary inequality is applied here to obtain a more general inequality for convex sequences.

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1. INTRODUCTION.

Let a and b be non-negative. Then the following elementary inequality was proved in [1].

\[
\frac{1}{n+1} [a^n + a^{n-1} b + \ldots + b^n] \geq \left(\frac{a+b}{2}\right)^n \quad (n=0,1,2,\ldots) \ldots \quad (1.1)
\]

Now this inequality can be obtained at once by taking \( f(t) = t^n \) in the well-known result

\[
\frac{1}{b-a} \int_a^b f(t) dt \geq f\left(\frac{a+b}{2}\right) \ldots \ldots \quad (1.2)
\]

which holds whenever \( f \) is convex in \([a,b]\). However, the method used in [1] to obtain (1.1) is interesting and it is the purpose of the present note to show that it can be used to prove a considerably more general result about sequences. Indeed this more general result will have (1.2) as a consequence.

2. MAIN RESULTS.

A lemma which we shall use is the following

**LEMMA.** If

\[
\beta_0 \geq \beta_1 \geq \beta_2 \geq \ldots \geq \beta_m
\]

and

\[
\sum_{\nu=0}^{m} \alpha_{\nu} = 0
\]

then...
and if the ordering of the $a_v$ is such that each positive $a$ precedes all the negative ones, then

$$\sum_{v=0}^{m} a_v \beta_v \geq 0.$$ 

This lemma, which is easily proved, is not the one stated by Haber but, essentially, it is what he used. For with $b_\parallel$ defined as in [1]

$$(i = 0 \parallel 2, \ldots, \frac{[n]}{2}; n \text{ even})$$

we do not in fact have

$$\sum_{i=0}^{\frac{[n]}{2}} b_\parallel = 0$$

which is what is needed to apply the lemma quoted there.

Our result is the following.

**Theorem.** Let $\{u_\parallel\}_\parallel=0$ be a convex sequence. Then

$$\sum_{v=0}^{n} u_\parallel \geq \frac{1}{2^n} \sum_{v=0}^{n} \binom{n}{v} u_\parallel \ldots.$$ 

(2.1)

To see that (1.2) is a consequence of (2.1) let the function $f(x)$ be bounded and convex (and hence continuous) on $[a, b]$ and take

$$u_\parallel = f(a + \frac{v}{n}(b-a)).$$

Then (2.1) reads

$$\frac{1}{n+1} \sum_{n=0}^{\binom{n}{v}} f(a + \frac{v}{n}(b-a)) \geq \frac{1}{2^n} \sum_{v=0}^{n} \binom{n}{v} f(a + \frac{v}{n}(b-a))\ldots$$ 

(2.2)

On letting $n \to \infty$ the left-hand side here tends to the left-hand side of (1.2). And by virtue of Bernstein's result

$$\lim_{n \to \infty} \sum_{v=0}^{\binom{n}{v}} \phi \left( \frac{v}{n} \right) x^{(1-v)}(1-x)^{v} = \phi(x) \ldots$$ 

(2.3)

whenever $\phi \in C[0,1]$ we see that the right-hand side of (2.2) tends to $f(a+b)$.

Merely take $\phi(x) = f(a + x(b-a))$ and $x = 1/2$ in (2.3).

We now proceed to prove (2.1).

**Proof.** Following Haber let us put $Q = \frac{[n]}{2}$ and write

$$\sum_{v=0}^{Q} \gamma_v = \left\{ \begin{array}{ll}
\gamma_0 + \gamma_1 + \ldots \ldots \ldots + \gamma_Q & \text{if } n \text{ is odd} \\
\gamma_0 + \gamma_1 + \ldots + \gamma_{Q-1} + \frac{1}{2} \gamma_Q & \text{if } n \text{ is even}.
\end{array} \right.$$

Then

$$\frac{1}{n+1} \sum_{v=0}^{n} u_\parallel - \frac{1}{2^n} \sum_{v=0}^{n} \binom{n}{v} u_\parallel = \sum_{v=0}^{Q} c_{\parallel}[u_\parallel + u_{n-v}].$$
where

\[ c_v = \frac{1}{n+1} - \frac{1}{2R^n} \]

Since \( \{u_v^0\} \) is convex then

\[ u_{v+1} + u_{n-v-1} \leq u_v + u_{n-v} \quad (0 \leq v \leq n-1) \]

which is to say that the sequence \( \{u_v + u_{n-v}\} \) is non-increasing. We see too that the sequence \( \{c_v\} \) is non-increasing and that \( \sum_{v=0}^{n} c_v = 0 \). Appealing to the Lemma quoted above we find that

\[ \sum_{v=0}^{n} c_v [u_v + u_{n-v}] \geq 0 \]

and this completes the proof of (2.1).

In conclusion I wish to thank the referee for his helpful advice concerning the lemma used here.

REFERENCES


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