AN AMENABILITY PROPERTY OF ALGEBRAS OF FUNCTIONS ON SEMIDIRECT PRODUCTS OF SEMIGROUPS

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ABSTRACT. Let S_1 and S_2 be semitopological semigroups, $S_1 \hat{\tau} S_2$ a semidirect product. An amenability property is established for algebras of functions on $S_1 \hat{\tau} S_2$. This result is used to decompose the kernel of the weakly almost periodic compactification of $S_1 \hat{\tau} S_2$ into a semidirect product.

KEY WORDS AND PHRASES. *Semitopological semigroup, semidirect product, compactification, amenability, strongly almost periodic, weakly almost periodic, kernel.*

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1. INTRODUCTION. Let S_1, S_2 be semitopological semigroups (in the terminology of Berglund and Hofmann (1)) with identities, each denoted by 1. That is S_1 and S_2 have (Hausdorff) topologies relative to which multiplication in S_1 and S_2 is separately continuous.

Let $\tau : S_2 \times S_1 \rightarrow S_1$ be a separately continuous map satisfying for each $s_1, t_1 \in S_1, s_2, t_2 \in S_2, \tau(s_2, s_1 t_1) = \tau(s_2, s_1) \tau(s_2, t_1), \tau(s_2 t_2, t_1) = \tau(s_2, \tau(t_2, t_1)), \tau(s_2, 1) = 1$, and $\tau(1, \cdot)$ is the identity map.

We shall assume the map $(s_1, s_2) \mapsto s_1 \tau(s_2, t_1) : S_1 \times S_2 \rightarrow S_1$ is continuous for each $t_1 \in S_1$. The semidirect product $S_1 \hat{\tau} S_2$ of S_1 and S_2 is the topological space

$S_1 \times S_2$ equipped with multiplication $(s_1, s_2)(t_1, t_2) = (s_1 \tau(s_2, t_1), s_2 t_2)$.

The above conditions on τ imply that $S_1 \hat{\tau} S_2$ is a semitopological semigroup with identity $(1, 1)$.

Let F be a closed translation invariant sub-C*-algebra of $C(S_1 \hat{\tau} S_2)$ (see §2 below) containing the constant functions. In previous papers (2) and (3), the author has formulated the necessary and sufficient conditions for the decomposition of the F -compactification of $(S_1 \hat{\tau} S_2)$, into a semidirect product. The decomposition may be written symbolically as

$$(S_1 \hat{\tau} S_2)^F = S_1^G \hat{\otimes} S_2^H \quad (1.1)$$

where $G = \{f(\cdot, 1) : f \in F\}$ and $H = \{f(1, \cdot) : f \in F\}$ and equality denotes canonical isomorphism (ρ being another semidirect product).

Applications of this decomposition were then made to the almost periodic (AP), strongly almost periodic (SAP) and left-uniformly continuous (LUC) cases. The situation is less well-behaved in the weakly almost periodic (WAP) case. For example, if $S_1 = S_2$ is any commutative topological semigroup with identity for which $WAP(S_1) = AP(S_1)$, then (1.1) fails even if $S_1 \hat{\tau} S_2$ is taken to be the special case of a direct product (Junghenn (4)).

However, in the present paper we shall prove an amenability property of algebras of functions on $S_1 \hat{\tau} S_2$ which generalizes a result of Junghenn (5) and provides conditions under which the kernel of the WAP-compactification of $S_1 \hat{\tau} S_2$ can be decomposed into a semidirect product.

2. PRELIMINARIES. Throughout this section S denotes a semitopological semigroup and $C(S)$ the C*-algebra of bounded continuous complex-valued functions on S . We define operators R_t and L_s on $C(S)$ by

$$R_t f(s) = f(st) = L_s f(t) \quad (s, t \in S ; f \in C(S))$$

Let F be a conjugate closed, norm closed linear subspace of $C(S)$ containing the constant function 1. Then F is *right* (resp. *left*) *translation invariant* if $R_s F \subset F$ (resp. $L_s F \subset F$) ; *translation invariant* if it is both left and right translation invariant.

A *mean* on F is a positive linear functional μ in F^* , the dual of F , such that $\mu(1) = 1 = \|\mu\|$. We denote by $M(F)$ the set of all means on F . A mean μ

on F is *multiplicative* if $\mu(fg) = \mu(f)\mu(g)$, $f, g \in F$. We denote the set of all multiplicative means on F by $MM(F)$.

If F is left (resp. right) translation invariant, then a mean μ is *left* (resp. *right*) *invariant* if, for each $f \in F$, $s \in S$, we have $\mu(L_s f) = \mu(f)$ (resp. $\mu(R_s f) = \mu(f)$). The set of all left (resp. right) invariant means on F shall be denoted by $LIM(F)$ (resp. $RIM(F)$). F is *left* (resp. *right*) *amenable* if $LIM(F) \neq \emptyset$ (resp. $RIM(F) \neq \emptyset$). If F is translation invariant and both left and right amenable, F is called *amenable*.

Now suppose F is left translation invariant. For each $\nu \in F^*$ define $T_\nu : F \rightarrow C(S)$ by $(T_\nu f)(s) = \nu(L_s f)$, $f \in F$, $s \in S$. Then F is *left introverted* if $T_\nu F \subset F$ for each $\nu \in M(F)$. If F is an algebra, then F is *left-m-introverted* if $T_\nu F \subset F$ for each $\nu \in MM(F)$. Right introversion and right-m-introversion are defined in an analogous manner.

If F is a sub- C^* -algebra of $C(S)$ then S^F denotes the spectrum (=space of nonzero continuous complex homomorphisms) of F equipped with the relativized weak* topology, and $e: S \rightarrow S^F$ the evaluation mapping.

If F is admissible (i.e. F is translation invariant, left-m-introverted, containing the constant functions) then a binary operation $(x, y) \rightarrow xy$ may be defined on S^F relative to which the pair (S^F, e) has the following properties:

- (i) S^F is a compact Hausdorff topological space and a semigroup such that for each $y \in S^F$, the mapping $x \rightarrow xy : S^F \rightarrow S^F$ is continuous;
- (ii) $e : S \rightarrow S^F$ is a continuous homomorphism with range dense in S^F such that for each $s \in S$, the mapping $x \rightarrow e(s)x : S^F \rightarrow S^F$ is continuous; and
- (iii) $e^* C(S^F) = F$.

The pair (S^F, e) is the *canonical F -compactification* of S .

Let $K(S)$, called the *kernel* of S , denote the minimal ideal of S . We shall use the amenability property in the next section to decompose the kernel of the WAP-compactification of $S_1 \oplus S_2$ into a semidirect product.

3. THE AMENABILITY THEOREM. Let S_1 and S_2 denote semitopological semigroups with identities and $S_1 \oplus S_2$ a semidirect product as defined in §1. We shall denote by

$q_1 : S_1 \rightarrow S_1 \oplus S_2$ and $q_2 : S_2 \rightarrow S_1 \oplus S_2$ the injection mappings ($q_1(s_1) = (s_1, 1)$, $q_2(s_2) = (1, s_2)$, for $s_1 \in S_1$, $s_2 \in S_2$). Let $q_i^* : C(S_1 \oplus S_2) \rightarrow C(S_i)$ denote the dual mapping of q_i , $i = 1, 2$.

THEOREM 3.1

(a) Suppose F is a left translation invariant, left introverted closed subspace of $C(S_1 \oplus S_2)$ containing the constant functions, and the semigroup

$D = \{s_2 \in S_2 : \tau(s_2, S_1) = S_1\}$ is dense in S_2 . Then F is left amenable if $q_1^* F$ and $q_2^* F$ are left amenable.

(b) Suppose F is a right translation invariant, right introverted closed subspace of $C(S_1 \oplus S_2)$ containing the constant functions. Then F is right amenable if $q_1^* F$ and $q_2^* F$ are right amenable.

PROOF. To prove (a) choose any $\mu_1 \in \text{LIM}(q_1^* F)$, and for each $f \in F$ define

$$(Uf)(s_2) = \mu_1(q_1^*(L_{(1, s_2)} f)), \quad s_2 \in S_2.$$

Then $U : F \rightarrow q_2^* F$. For let $f \in F$. Since F is left introverted,

$$T_v F \subset F, \quad \forall v \in M(F) \text{ where } (T_v f)(s_1, s_2) = v(L_{(s_1, s_2)} f), \quad f \in F, \quad (s_1, s_2) \in (S_1 \oplus S_2).$$

Observe that

$$\begin{aligned} (Uf)(s_2) &= \mu_1(q_1^*(L_{(1, s_2)} f)) = T_{(\mu_1 \circ q_1^*)} f(1, s_2) \\ &= q_2^*(T_{(\mu_1 \circ q_1^*)} f)(s_2), \quad \text{for any } s_2 \in S_2. \end{aligned}$$

Then $UF \in q_2^* F$ since $T_{(\mu_1 \circ q_1^*)} f \in F$. Furthermore, $U : F \rightarrow q_2^* F$ is a positive linear operator of norm 1 since μ_1 is a mean on $q_1^* F$.

Let $\mu_2 \in \text{LIM}(q_2^* F)$ and put $\mu = \mu_2 \circ U$. Then $\mu \in F^*$, $\mu(f) \geq 0$ for each $f \geq 0$ in F , and $\mu(1) = 1$. Thus μ is a mean on F .

We must show $\mu \in \text{LIM}(F)$. Observe that for $s_1 \in S_1$, $s_2 \in S_2$,

$$\begin{aligned} q_1^*(L_{(s_1, 1)} L_{(1, s_2)} f) &= q_1^*(L_{(1, s_2)}(s_1, 1) f) \\ &= q_1^*(L_{(\tau(s_2, s_1), s_2)} f). \end{aligned} \quad (3.1)$$

Furthermore, for any $g \in F$, $s_1, t_1 \in S_1$,

$$\begin{aligned} q_1^*(L_{(s_1, 1)} g)(t_1) &= L_{(s_1, 1)} g(t_1, 1) = g(s_1 t_1, 1) \\ &= (q_1^* g)(s_1 t_1) = L_{s_1}(q_1^* g)(t_1). \end{aligned}$$

Thus,

$$q_1^*(L_{(s_1, 1)} L_{(1, s_2)} f) = L_{s_1}(q_1^* L_{(1, s_2)} f). \quad (3.2)$$

By (3.1) and (3.2) we obtain for $d \in D$, $s_1 \in S_1$, $f \in F$,

$$\begin{aligned} \mu_1(q_1^{*(L_{(\tau(d, s_1), d)} f)}) &= \mu_1(q_1^{*(L_{(s_1, 1)} L_{(1, d)} f)}) \\ &= \mu(L_{s_1} q_1^{*(L_{(1, d)} f)}) = \mu_1(q_1^{*(L_{(1, d)} f)}) \\ &= (Uf)(d). \end{aligned} \quad (3.3)$$

By the definition of D and the continuity in the variable s_1 of the extreme left side of (3.3), we obtain,

$$\mu_1(q_1^{*(L_{(s_1, d)} f)}) = (Uf)(d) \quad (d \in D, s_1 \in S_1).$$

Since $\bar{D} = S_2$ we therefore have

$$\mu_1(q_1^{*(L_{(s_1, s_2)} f)}) = (Uf)(s_2) \quad (s_1 \in S_1, s_2 \in S_2).$$

That is,

$$UL_{(s_1, 1)} f = Uf, \quad \forall s_1 \in S_1. \quad (3.4)$$

Observe that for $s_2, t_2 \in S_2$,

$$\begin{aligned} U(L_{(1, s_2)} f)(t_2) &= \mu_1(q_1^{*(L_{(1, t_2)} L_{(1, s_2)} f)}) \\ &= \mu_1(q_1^{*(L_{(1, s_2 t_2)} f)}) = (Uf)(s_2 t_2) \\ &= (L_{s_2} Uf)(t_2). \end{aligned}$$

Thus,

$$U(L_{(1, s_2)} f) = L_{s_2} Uf, \quad s_2 \in S_2. \quad (3.5)$$

By (3.4) and (3.5) we obtain for any $s_1 \in S_1$, $s_2 \in S_2$

$$\begin{aligned} \mu(L_{(s_1, s_2)} f) &= \mu(L_{(1, s_2)} L_{(s_1, 1)} f) \\ &= \mu_2[U(L_{(1, s_2)} L_{(s_1, 1)} f)] \\ &= \mu_2[L_{s_2} U(L_{(s_1, 1)} f)] \\ &= \mu_2(L_{s_2} Uf) = \mu_2(Uf) = \mu(f). \end{aligned}$$

Thus $\mu \in \text{LIM}(F)$ and we are done.

The proof of (b) is done in an analogous manner and is, in fact, much easier.

Choose any $\mu_1 \in \text{RIM}(q_1^* F)$ and for each $f \in F$, define

$$(Uf)(s_2) = \mu_1(q_1^{*(R_{(1, s_2)} f)}), \quad s_2 \in S_2.$$

Then $U : F \rightarrow q_2^* F$ since F is right introverted. Furthermore, $U : F \rightarrow q_2^* F$ is a positive linear operator of norm 1 since μ_1 is a mean on $q_1^* F$. Let $\mu_2 \in \text{RIM}(q_2^* F)$

and put $\mu = \mu_2 \cdot U$. Then μ is a mean on F and we must show $\mu \in \text{RIM}(F)$.

Observe that for any $s_1, t_1 \in S_1, s_2, t_2 \in S_2, f \in F$,

$$\begin{aligned} q_1^{*R}(1, t_2)^{R(s_1, s_2)} f(t_1) &= f[(t_1, 1)(1, t_2)(s_1, s_2)] \\ &= f[(t_1 \tau(t_2, s_1), t_2 s_2)] \\ &= f[(t_1 \tau(t_2, s_1), 1)(1, t_2 s_2)] \\ &= q_1^{*R}(1, t_2 s_2)^{f(t_1 \tau(t_2, s_1))} \\ &= R_{\tau(t_2, s_1)} q_1^{*R}(1, t_2 s_2)^{f(t_1)}. \end{aligned}$$

Thus, $q_1^{*R}(1, t_2)^{R(s_1, s_2)} f = R_{\tau(t_2, s_1)} q_1^{*R}(1, t_2 s_2)^f$,

and therefore since $\mu_1 \in \text{RIM}(q_1^* F)$,

$$\begin{aligned} U(R(s_1, s_2)^f)(t_2) &= \mu_1(q_1^{*R}(1, t_2)^{R(s_1, s_2)} f) \\ &= \mu_1(R_{\tau(t_2, s_1)} q_1^{*R}(1, t_2 s_2)^f) = \mu_1(q_1^{*R}(1, t_2 s_2)^f) \\ &= R_{s_2} Uf(t_2). \end{aligned}$$

Then,

$$\begin{aligned} \mu(R(s_1, s_2)^f) &= \mu_2[U(R(s_1, s_2)^f)] = \mu_2(R_{s_2} Uf) \\ &= \mu_2(Uf) = \mu(f). \end{aligned}$$

Hence $\mu \in \text{RIM}(F)$ and we are done.

4. Application to $K(S_1 \hat{\otimes} S_2)^{\text{WAP}}$

Let S be a semitopological semigroup and let $\text{SAP}(S)$ denote the closed linear span in $C(S)$ of the coefficients of all finite-dimensional continuous unitary representation of S . $\text{SAP}(S)$ is called the space of *strongly almost periodic* functions on S . Let $\text{WAP}(S) = \{f \in C(S) : R_S f \text{ is relatively weakly compact}\}$. $\text{WAP}(S)$ is called the space of *weakly almost periodic* functions on S . (See Berglund, Junghenn and Milnes (6) for properties of $\text{SAP}(S)$, $\text{WAP}(S)$).

In (3) it was shown that if S_1, S_2 are semitopological semigroups with identities then

$$(S_1 \hat{\otimes} S_2)^{\text{SAP}} = X \hat{\otimes} S_2^{\text{SAP}} \quad (4.1)$$

where $(S_1 \hat{\otimes} S_2)^{\text{SAP}}$ and S_2^{SAP} are the canonical SAP -compactifications of $S_1 \hat{\otimes} S_2$ and S_2 respectively, X is a compact topological group which is a homomorphic image

of the canonical SAP-compactification of S_1 , and equality denotes canonical isomorphism.

We now prove the following lemma which shall be used in the decomposition of the kernel.

LEMMA. 4.1 Let S be a semitopological semigroup such that $\text{WAP}(S)$ is amenable. Let (S^{WAP}, Ψ) be a WAP-compactification of S , ξ the identity of $K(S^{\text{WAP}})$, $\rho_\xi: S^{\text{WAP}} \rightarrow K(S^{\text{WAP}})$ be right multiplication. Then $(K(S^{\text{WAP}}), \rho_\xi^\Psi)$ is an SAP-compactification of S .

PROOF. Since $\text{WAP}(S)$ is amenable, $K(S^{\text{WAP}}) = S^{\text{WAP}}_\xi$ and is a compact topological group (deLeeuw and Glicksberg (7)). Then ρ_ξ maps S^{WAP} onto $K(S^{\text{WAP}})$ and $\rho_\xi^\Psi: S \rightarrow K(S^{\text{WAP}}) = S^{\text{WAP}}_\xi$ is a continuous homomorphism with range dense in S^{WAP}_ξ . Observe that $\rho_\xi|_{K(S^{\text{WAP}})}$ is the identity mapping on $K(S^{\text{WAP}})$.

Let (S^{SAP}, γ) be the canonical SAP-compactification of S . By the universal mapping property of SAP and WAP compactifications, there exist continuous homomorphisms ϕ and θ such that $\phi: S^{\text{SAP}} \rightarrow K(S^{\text{WAP}})$, $\theta: S^{\text{WAP}} \rightarrow S^{\text{SAP}}$ and $\phi\gamma = \rho_\xi^\Psi, \theta\Psi = \gamma$.

Observe further that since $\phi\theta\Psi = \rho_\xi^\Psi$, then $\phi\theta = \rho_\xi$ by the continuity of $\phi\theta$, ρ_ξ and the fact that $\overline{\Psi(S)} = S^{\text{WAP}}$.

All of the above relations are illustrated in the following commutative diagram:

It suffices to show that ϕ is one-to-one so that $K(S^{\text{WAP}})$ will be an SAP-compactification of S . Let $y_1, y_2 \in S^{\text{SAP}}$. Then there exist $x_1, x_2 \in S^{\text{WAP}}$ such that $\theta(x_1) = y_1$ and $\theta(x_2) = y_2$. Suppose $\phi(y_1) = \phi(y_2)$. Then $\phi(\theta(x_1)) = \phi(\theta(x_2))$. Since S^{SAP} is a compact topological group and $\theta(\xi)$ is an idempotent in S^{SAP} , $\theta(\xi)$ is the identity of S^{SAP} . Thus $\theta(x_1) = \theta(x_1)\theta(\xi) = \theta(x_1\xi)$ ($i=1,2$), so $\phi(\theta(x_1\xi)) = \phi(\theta(x_2\xi))$. On the other hand,

$$\phi(\theta(x_i \xi)) = \rho_\xi(x_i \xi) = x_i \xi \quad (i = 1, 2),$$

so, $x_1 \xi = x_2 \xi$, and hence $y_1 = \theta(x_1 \xi) = \theta(x_2 \xi) = y_2$. //

We shall use the relation (4.1), the above lemma and the results in the following discussion to establish conditions under which we may express

$K[(S_1 \oplus S_2)^{\text{WAP}}]$ as a semidirect product

$$K[(S_1 \oplus S_2)^{\text{WAP}}] = X \oplus K(S_2^{\text{WAP}})$$

where equality denotes canonical isomorphism and X is a compact topological group.

We shall assume that $\text{WAP}(S_1)$ and $\text{WAP}(S_2)$ are amenable. By (deLeeuw and Glicksberg (7), Lemma 5.2) since $q_i : S_i \rightarrow S_1 \oplus S_2$ is a continuous homomorphism for $i = 1, 2$, then $F_1 = q_1^* \text{WAP}(S_1 \oplus S_2) \subset \text{WAP}(S_1)$, and $F_2 = q_2^* \text{WAP}(S_1 \oplus S_2) \subset \text{WAP}(S_2)$. (In fact, equality holds in the latter.) Thus, $q_1^* \text{WAP}(S_1 \oplus S_2)$ and $q_2^* \text{WAP}(S_1 \oplus S_2)$ are amenable and if we assume $D = \{s_2 \in S_2 : \tau(\overline{s_2, S_1}) = S_2\}$ is dense in S_1 , then $\text{WAP}(S_1 \oplus S_2)$ is amenable by Theorem 3.1.

By ((7), Theorem 4.11) $K[(S_1 \oplus S_2)^{\text{WAP}}]$ and $K(S_2^{\text{WAP}})$ are compact topological groups. Furthermore, by Lemma 4.1, $K[(S_1 \oplus S_2)^{\text{WAP}}]$ is a SAP-compactification of $S_1 \oplus S_2$, and $K(S_2^{\text{WAP}})$ is a SAP-compactification of S_2 (symbollically denoted by $K[(S_1 \oplus S_2)^{\text{WAP}}] = (S_1 \oplus S_2)^{\text{SAP}}$, and $K(S_2^{\text{WAP}}) = S_2^{\text{SAP}}$, respectively, where equality denotes canonical isomorphism). Thus, we have proved the following

PROPOSITION 4.2 Let S_1, S_2 be semitopological semigroups with identities and $S_1 \oplus S_2$ a semidirect product. Suppose $\text{WAP}(S_1), \text{WAP}(S_2)$ are amenable, and $D = \{s_2 \in S_2 : \tau(\overline{s_2, S_1}) = S_1\}$ is dense in S_2 . Then $\text{WAP}(S_1 \oplus S_2)$ is amenable. Furthermore, we may represent $K[(S_1 \oplus S_2)^{\text{WAP}}]$ as a semidirect product $K[(S_1 \oplus S_2)^{\text{WAP}}] = X \oplus K(S_2^{\text{WAP}})$, where equality denotes canonical isomorphism, $(S_1 \oplus S_2)^{\text{WAP}}$ and S_2^{WAP} are canonical WAP-compactifications of $S_1 \oplus S_2$ and S_2 respectively, and X is a compact topological group which is a continuous homomorphic image of the canonical SAP-compactification of S_1 . //

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