KNOTS WITH PROPERTY $R+$

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ABSTRACT. If we consider the set of manifolds that can be obtained by surgery on a fixed knot K , then we have an associated set of numbers corresponding to the Heegaard genus of these manifolds. It is known that there is an upper bound to this set of numbers. A knot K is said to have Property $R+$ if longitudinal surgery yields a manifold of highest possible Heegaard genus among those obtainable by surgery on K . In this paper we show that torus knots, 2-bridge knots, and knots which are the connected sum of arbitrarily many $(2,m)$ -torus knots have Property $R+$. It is shown that if K is constructed from the tangles $(B_1, t_1), (B_2, t_2), \dots, (B_n, t_n)$, then $T(K) \leq 1 + \sum_{i=1}^n T(B_i, t_i)$ where $T(K)$ is the tunnel of K and $T(B_i, t_i)$ is the tunnel number of the tangle (B_i, t_i) . We show that there exist prime knots of arbitrarily high tunnel number that have Property $R+$ and that manifolds of arbitrarily high Heegaard genus can be obtained by surgery on prime knots.

KEY WORDS AND PHRASES. Knot, surgery, Heegaard genus, tangle.

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1. INTRODUCTION.

A traditional method of constructing 3-manifolds is to perform Dehn surgery on a knot or link in S^3 . As a result of this relationship between knots and 3-manifolds, several negatively defined properties of knots, namely Property P

and Property R, have been studied. The purpose of this paper is to introduce a positively stated property, Property R_+ . This property is clearly related to Property R in that it is at least as strong and generally stronger than Property R. If the knot K should have Property R_+ it would mean that trivial surgery and longitudinal surgery yield respectively the least complex and the most complex 3-manifolds obtainable by surgery on K in terms of Heegaard genus. We shall demonstrate that infinitely many knots have Property R_+ .

If X is a point set, we shall use $\text{cl}(X)$ for the closure of X , $\text{int}(X)$ for the interior of X and ∂X for the boundary of X . If K is a cube-with-knotted hole, the longitudinal curve of K will be the simple closed curve on ∂K , unique up to isotopy, which bounds an orientable surface in K . The genus of a 3-manifold is defined to be the minimal genus of a Heegaard splitting of the manifold. If X is a polyhedron contained in the P. L. 3-manifold M , then $N(X) \subset M$ is called a regular neighborhood of X in M if $X \subset N(X)$ and $N(X)$ is a 3-manifold which can be simplicially collapsed to X . This paper deals with P. L. topology. Therefore, all manifolds in this paper are assumed to be simplicial and all maps to be piecewise linear.

2. PROPERTY R_+ .

Let $K \subset S^3$ be a knot. If we consider the set of manifolds that can be obtained by surgery on K , then of course we will have a set of numbers corresponding to the Heegaard genus of these manifolds. We know by [2] that there is an upper bound to this set of numbers.

DEFINITION. K is said to have Property R_+ if and only if longitudinal surgery on K yields a manifold of maximal Heegaard genus among those that can be obtained by surgery on K .

PROPOSITION. All two bridge knots, all torus knots, and all knots which are the connected sum of arbitrarily many $(2, m)$ -torus knots have Property R_+ .

PROOF. We know by [10] that all two bridge knots have Property R . Hence by [2] these knots have Property $R+$. As for torus knots, we know by [8] that all torus knots have Property R . Thus by [1] we know that torus knots have Property $R+$. Finally, we know by [3] that the connected sum of arbitrarily many $(2, m)$ -torus knots will be a knot which also has Property $R+$.

While it is true that a knot of arbitrarily high complexity can be fashioned from the connected sum of $(2, m)$ -torus knots, it would be more interesting to find a collection of prime knots of arbitrarily high complexity that also have Property $R+$. In order to find such a collection of knots we must first consider the theory of tangles and develop a concept of tunnel number for tangles.

3. THE TUNNEL NUMBER OF TANGLES.

The original concept of a tangle was developed by Conway in [4]. We shall use the somewhat modified definition of a tangle as given by Kirby and Lickorish in [6].

DEFINITION. A tangle is a pair (B, t) where B is a 3-cell and t is a pair of disjoint arcs in B such that $t \cap \partial B = \partial t$. The tangles (B_1, t_1) and (B_2, t_2) are said to be equivalent if there is a homeomorphism of pairs between (B_1, t_1) and (B_2, t_2) . A tangle is trivial if it is equivalent to $(D \times I, \{x, y\} \times I)$ where D is a disk with $\{x, y\} \subset \text{int } D$. An arc $A \subset \text{int } B$ with $A \cap t = \partial A$ will be called a tunnel.

If t_1 and t_2 are the two arcs making up $t \subset B$, we note that up to isotopy there are only two ways of adding disjoint arcs a_1 and a_2 with $a_1 \cup a_2 \subset \partial B$ and each arc connecting t_1 to t_2 . Obviously, $t_1 \cup a_1 \cup t_2 \cup a_2$ will be a knot $K \subset S^3$. It is then possible to add a set of pairwise disjoint tunnels $\{A_1, A_2, \dots, A_n\}$ so that:

- (i) $H = N(K \cup A_1 \cup A_2 \cup \dots \cup A_n)$ bounds $n + 1$ pairwise disjoint disks $\{D_1, D_2, \dots, D_{n+1}\}$ with $\text{int } D_i \subset \text{int } B$;
- (ii) $N(H \cup D_1 \cup D_2 \cup \dots \cup D_{n+1})$ is a spanning 3-cell in B that contains a spanning unknotted arc.

To see that this is always possible, one merely needs to look at a regular projection of (B, t) and add a tunnel at each double point of the projection.

The tunnel number of K relative to (B, t) , $T(K, (B, t))$, is the smallest number of tunnels that need to be added to K in order to satisfy both (i) and (ii) above. It should be noted that in general $T(K, (B, t))$ will be a larger number than the tunnel number $T(K)$ as defined in [2]. Let K_1 and K_2 be the two knots that can be formed by adding arcs a_1 and a_2 in ∂B to $t \subset B$. As a rule $T(K_1, (B, t))$ and $T(K_2, (B, t))$ will be different numbers. For example, consider the tangle (B, t) in Figure 1. One way to complete t to a knot yields the square knot K_1 . Since the square knot has the same fundamental group as a granny knot, we know by [3] that K_1 has tunnel number 2. Hence $T(K_1, (B, t))$ is also 2. The other completion of t , K_2 , is the twist knot 6_1 . Since 6_1 is a 2-bridge knot, 6_1 has tunnel number 1 and hence $T(K_2, (B, t)) = 1$.

DEFINITION. $T((B, t)) = \max \{T(K_1, (B, t)), T(K_2, (B, t))\}$ where $T((B, t))$ is the tunnel number of the tangle (B, t) .

If (B_1, t_1) and (B_2, t_2) are tangles, it is possible to create a new tangle called the partial sum of (B_1, t_1) and (B_2, t_2) by identifying a (disk, point pair) in the boundary of one tangle with a (disk, point pair) in the boundary of the other tangle. Any of the different ways that this can be accomplished will be denoted $(B_1, t_1) + (B_2, t_2)$. If (S^3, K) is the result of identifying $\partial(B_1, t_1)$ to $\partial(B_2, t_2)$ by a homeomorphism h , the result will be denoted as $(B_1, t_1) \cup_h (B_2, t_2)$. Therefore, if $(B_1, t_1), (B_2, t_2), \dots, (B_n, t_n)$ are tangles, then one can write $(S^3, K) = (\sum_{i=1}^{n-1} (B_i, t_i)) \cup_h (B_n, t_n)$ where $K \subset S^3$ is any one of the infinitely many knots that can be created in this fashion from the tangles $(B_1, t_1), (B_2, t_2), \dots, (B_n, t_n)$.

THEOREM. If $(S^3, K) = (\sum_{i=1}^{n-1} (B_i, t_i)) \cup_h (B_n, t_n)$ then $T(K) \leq 1 + \sum_{i=1}^n T((B_i, t_i))$.

PROOF. Within each 3-cell B_i we add the appropriate number of tunnels so that we can find unknotted spanning arcs a_{1i} and a_{2i} (possibly not disjoint) with

$\partial(a_{1i} \cup a_{2i}) = \partial t_i$ and $a_{1i} \cup a_{2i} \subset t_i \cup A_1 \cup A_2 \cup \dots \cup A_n$. Thus $(B_i, a_{1i} \cup a_{2i})$ is equivalent to a trivial (possibly pinched) tangle. Hence both $(B_n, a_{1n} \cup a_{2n})$ and $(\sum_{i=1}^{n-1} (B_i, a_{1i} \cup a_{2i}))$ are trivial tangles, and $(S^3, \bigcup_{i=1}^n (a_{1i} \cup a_{2i}))$ may be at most a 2-bridge knot pair. Therefore, the addition of at most one more tunnel A , will yield a 1-complex C such that both $N(C)$ and $cl(S^3 - N(C))$ are handlebodies.

COROLLARY. If K is obtained by adding the tangles $(B_1, t_1), (B_2, t_2), \dots, (B_n, t_n)$ and M is obtained by Dehn surgery on K , then $H(M) \leq 2 + \sum_{i=1}^n T((B_i, t_i))$ where $H(M)$ is the Heegaard genus of M .

We note that the formula given in the above theorem is the best possible such formula. If both (B_1, t_1) and (B_2, t_2) are trivial tangles, then $T(B_1, t_1) = T(B_2, t_2) = 0$. Yet it is possible to construct a knot K from $t_1 \cup t_2$ with $T(K) = 1$. On the other hand the square knot K can be obtained from the tangle (B, t) in Figure 1 by adding a trivial tangle. But $T(K) = T((B, t))$.

4. A COLLECTION OF PRIME KNOTS WITH PROPERTY $R+$.

In [6], Kirby and Lickorish pointed out a special class of tangles which they called prime tangles. The tangle (B, t) is said to be prime if and only if every 2-sphere in B which meets t transversely in two points, bounds in B a 3-cell meeting t in an unknotted spanning arc and no properly embedded disk in B separates the arcs of t .

In [7] Lickorish proved that if $(S^3, K) = (\sum_{i=1}^{n-1} (B_i, t_i)) \cup_h (B_n, t_n)$ where $n \geq 2$ and (B_i, t_i) is a prime tangle for $1 \leq i \leq n$, then K is a prime knot. He also demonstrated that the tangle shown in Figure 1 is a prime tangle. We shall be using these results in the formation of our collection of prime knots.

Let K_n be the knot formed from n prime tangles as shown in Figure 2. As we've seen before, K_1 is the prime knot 6_1 . K_n for $n \geq 2$ satisfies the hypothesis of Lickorish's theorem and hence is also a prime knot. As we saw in Section 3, each prime tangle used in the construction of K_n has tunnel number 2. Therefore,

$T(K_n) \leq 2n + 1$. The half twist at the top of K_n in Figure 2 yields a knot which needs only one tunnel in its top tangle and no additional tunnel from the addition of the trivial tangle to complete (S^3, K_n) . Therefore $T(K_n) \leq 2n - 1$. The placement of these $2n - 1$ tunnels is shown in Figure 3. If A_i is the i -th tunnel added to K_n , then clearly $N(K_n \cup A_1 \cup A_2 \cup \dots \cup A_{2n-1})$ is a handlebody with $cl(S^3 - N(K_n \cup A_1 \cup A_2 \cup \dots \cup A_{2n-1}))$ also a handlebody.

A Wirtinger presentation of the fundamental group of $S^3 - K_n$ is as follows:

$$\{a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2, \dots, a_n, b_n, c_n, d_n, e_n, f_n, a_{n+1}, b_{n+1} \mid R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}, R_{21}, R_{22}, R_{23}, R_{24}, R_{25}, R_{26}, \dots, R_{n1}, R_{n2}, R_{n3}, R_{n4}, R_{n5}, R_{n6}, a_1 b_{n+1}^{-1}\}, \text{ where } R_{i1} = a_i d_i a_i^{-1} f_i^{-1}, R_{i2} = d_i a_i d_i^{-1} c_i^{-1}, R_{i3} = c_i b_i c_i^{-1} d_i^{-1}, R_{i4} = f_i a_{i+1} e_i^{-1} a_{i+1}^{-1}, R_{i5} = c_i e_i a_{i+1}^{-1} e_i^{-1}, \text{ and } R_{i6} = e_i c_i b_{i+1}^{-1} c_i^{-1} \text{ for } 1 \leq i \leq n.$$

This presentation can be simplified by use of Tietze transformations to:

$$\{d_1, a_2, d_2, a_3, d_3, \dots, a_n, d_n, a_{n+1} \mid \tilde{R}_{11}, \tilde{R}_{12}, \tilde{R}_{21}, \tilde{R}_{22}, \dots, \tilde{R}_{n-11}, \tilde{R}_{n-12}, \tilde{R}_{n1}\}$$

where

$$\tilde{R}_{i1} = d_i a_i d_i^{-1} a_{i+1}^{-1} a_i d_i a_i^{-1} a_{i+1}^{-1} a_i d_i^{-1} a_i^{-1} a_{i+1}^{-1} \quad \text{for } 1 \leq i \leq n$$

$$\tilde{R}_{i2} = a_{i+1}^{-1} = d_n a_n^{-1} d_n^{-1} a_{n+1}^{-1} a_n d_n a_n^{-1} a_{n+1}^{-1} d_n a_n d_n^{-1}.$$

Using the free calculus as developed in [5] we calculate the elementary ideals of K_n . For K_1 we find that $E_1 = (2t^2 - 5t + 2)$, and $E_m = (1)$ for $m \geq 2$. For K_n we find $E_{2n-1} = (t-2, 2t-1)$, and $E_m = (1)$ for $m \geq 2n$. Therefore $\pi_1(cl(S^3 - N(K_n)))$ is a $2n$ generator group and hence $T(K_n) = 2n - 1$.

As a result of [2], it is clear that $T(K) < br(K)$. For example, all torus knots have tunnel number 1 although there exist torus knots of arbitrarily high bridge number. Thus the tunnel number of a knot is a more strict measure of the complexity of a knot than is the bridge number.

THEOREM. There exist prime knots of arbitrarily high tunnel number that have Property $R+$.

PROOF. Let M_n denote the manifold obtained by longitudinal surgery on K_n .

The infinite cyclic covering of M_n has the same $\mathbb{Z}[t, t^{-1}]$ module-structure as the infinite cyclic cover of the K_n knot complement. This follows directly from the method of construction of such covers as demonstrated in [9]. Hence all of the Alexander invariants of M_n and the K_n knot complement are identical. Hence the Heegaard genus of M_n is at least $2n$. Since $T(K_n) = 2n - 1$, we know by [2] that any manifold obtained by surgery on K_n will have at most a Heegaard genus of $2n$. Therefore K_n has property $R+$.

COROLLARY. Manifolds of arbitrarily high Heegaard genus can be obtained by surgery on prime knots.

Figure 1

Figure 2

Figure 3

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