SUPREMUM NORM DIFFERENTIABILITY

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ABSTRACT. The points of Gateaux and Fréchet differentiability of the norm in $C(T, E)$ are obtained, where T is a locally compact Hausdorff space and E is a real Banach space. Applications of these results are given to the space $\ell_\infty(E)$ of all bounded sequences in E , and to the space $B(\ell_1, E)$ of all bounded linear operators from ℓ_1 into E .

KEY WORDS AND PHRASES. Banach spaces, continuous functions, vector-valued functions, supremum norm, smooth points.

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1. INTRODUCTION.

In [1], Banach proved that if T is a compact metric space and $C(T)$ is the Banach space of all continuous real valued functions on T , with the supremum norm, then

$$\lim_{\lambda \rightarrow 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda}$$

exists for all $g \in C(T)$ if and only if there exists a $t_0 \in T$ such that $|f(t_0)| > |f(t)|$ for all $t \in T$, $t \neq t_0$.

This theorem, however, is no longer true if T is a locally compact, non-compact, Hausdorff space; as can easily be seen by considering the Banach space ℓ_∞ of all bounded real valued sequences with the supremum norm.

In fact, if $\mathbb{N}$ is the set of positive integers equipped with the discrete topology, then $\ell_\infty = C(\mathbb{N})$, the space of all bounded continuous functions on $\mathbb{N}$. If we let $x = \{x_n\}_{n \geq 1} \in \ell_\infty$, where $x_1 = 1$ and $x_n = \frac{n-1}{n}$ for $n > 1$, then x peaks at $n = 1$, but because of the behaviour of x at infinity and the existence of Banach limits, it is possible to find two distinct support functionals to the ball in ℓ_∞ at x , so that x is not a smooth point.

In this note, we characterize the points of Gateaux and Fréchet differentiability of the norm function in $C(T, E)$, the space of all bounded continuous E -valued functions on the locally compact Hausdorff space T , where E is a real Banach space.

Two applications of these results are given. The first is to the space $\ell_\infty(E)$ of all bounded sequences in E , and the second to the space $B(\ell_1, E)$ of all bounded linear operators from ℓ_1 into E .

2. DEFINITIONS AND NOTATION.

In the following, E denotes a real Banach space and E^* denotes the dual of E . The unit ball of E is $B_E = \{x \in E \mid \|x\| \leq 1\}$ and its boundary $S_E = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of E .

A Banach space E is said to be smooth at $x \in E \sim \{0\}$ if and only if there exists a unique hyperplane of support to B_E at $\frac{x}{\|x\|}$; that is, there exists only one continuous linear functional $\phi \in E^*$ with $\|\phi\| = 1$ such that $\phi(x) = \|x\|$. Such a linear functional $\phi \in E^*$ is called the support functional to B_E at $\frac{x}{\|x\|}$, and $\phi^{-1}(\{1\})$ is called the hyperplane of support to B_E at $\frac{x}{\|x\|}$. A Banach space E is said to be a smooth Banach space if it is smooth at every $x \in S_E$.

The norm function $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is said to be Gateaux differentiable at $x \in E \sim \{0\}$ if and only if there exists a functional $\phi \in E^*$ with

$$\lim_{\lambda \rightarrow 0} \left| \frac{\|x + \lambda h\| - \|x\|}{\lambda} - \phi(h) \right| = 0, \quad (*)$$

for every $h \in E$. The functional ϕ is called the Gateaux derivative of the norm at $x \in E$. The norm function $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is said to be Fréchet differentiable at $x \in E \setminus \{0\}$ if and only if there exists a functional $\phi \in E^*$ such that

$$\lim_{\|h\| \rightarrow 0} \frac{|\|x+h\| - \|x\| - \phi(h)|}{\|h\|} = 0, \quad (**)$$

that is, the limit in (*) exists uniformly for $h \in B_E$.

It is well known, Mazur [8], that E is smooth at $x \in E \setminus \{0\}$ if and only if

$\lim_{\lambda \rightarrow 0} \frac{\|x + \lambda h\| - \|x\|}{\lambda}$ exists for all $h \in E$, if and only if the norm function $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is Gateaux differentiable at x .

3. SMOOTH POINTS IN $C(T, E)$.

If T is a topological space and E is a real Banach space, then $C(T, E)$ denotes the Banach space of all bounded continuous E -valued functions on T , with the supremum norm; that is,

$$C(T, E) = \{f: T \rightarrow E \mid f \text{ is bounded and continuous}\},$$

and $\|f\| = \sup \{\|f(t)\| : t \in T\}$, for $f \in C(T, E)$.

As mentioned earlier, Banach [1] proved that if T is a compact metric space, then $C(T) = C(T, \mathbb{R})$ is smooth at $f \neq 0$ if and only if f is a peaking function, that is, there exists a point $t_0 \in T$ such that $\|f\| = |f(t_0)| > |f(t)|$ for all $t \in T$, $t \neq t_0$.

Kondagunta [6], and Cox and Nadler [3], have characterized the points of Gateaux and Fréchet differentiability of the norm in $C(T, E)$ when T is compact Hausdorff. Cox and Nadler, in the same paper, give a characterization of the points of Fréchet differentiability of the norm in $C(T, \mathbb{R})$ when T is locally compact Hausdorff.

In this section, we generalize these results to the space $C(T, E)$ when T is locally compact Hausdorff. The techniques used by Cox and Nadler will not work in this case, since, in general, the range of $f \in C(T, E)$ is no longer relatively compact and hence no extension to an $\hat{f} \in C(\beta T, E)$ is possible. However, a slight modification of the argument given in Köthe [7], (page 352), for the corresponding result in ℓ_∞ , does work.

As usual, the results on smoothness are the more difficult, and the results on Fréchet differentiability follow as a corollary.

THEOREM 3.1. Let T be a locally compact Hausdorff space and E a Banach space. A point $f \in C(T, E)$, $f \neq 0$, is a smooth point of $C(T, E)$ if and only if

(i) there exists a $t_0 \in T$ such that $\|f(t_0)\| > \|f(t)\|$ for all $t \in T$,

$t \neq t_0$,

(ii) there exists a compact neighborhood K of t_0 such that

$$\sup_{t \in T \sim K} \|f(t)\| < \|f\|,$$

(iii) $f(t_0)$ is a smooth point of E .

PROOF.

A. Assume first that the norm $\|\cdot\| : C(T, E) \rightarrow \mathbb{R}^+$ is Gateaux differentiable at $f \in C(T, E)$, where we may (and do) assume that $\|f\| = 1$.

(I) We show first that if the mapping $\|f(\cdot)\| : T \rightarrow \mathbb{R}^+$ achieves its maximum at $t_0 \in T$, then t_0 is unique. Suppose there exist $t_0, t_1 \in T$, $t_0 \neq t_1$, such that $\|f(t_0)\| = \|f(t_1)\| = 1$. For $t \in T$ and $\phi \in E^*$, let $\delta_{\phi, t} \in C(T, E)^*$ denote the evaluation functional given by $\delta_{\phi, t}(g) = \phi(g(t))$ for $g \in C(T, E)$; then $\|\delta_{\phi, t}\| = 1$ for all $t \in T$, $\phi \in S_{E^*}$. Using the Hahn-Banach theorem, choose $\phi_0, \phi_1 \in E^*$ with $\|\phi_0\| = \|\phi_1\| = 1$ such that $\phi_0(f(t_0)) = \phi_1(f(t_1)) = 1$. Then δ_{ϕ_0, t_0} and δ_{ϕ_1, t_1} are distinct support functionals to the ball in $C(T, E)$ at f , which contradicts the fact that f is a smooth point in $C(T, E)$.

(II) We show next that given any compact set $K \subseteq T$, either

$\sup_{t \in T \sim K^\circ} \|f(t)\| < 1$, or there exists a $t_0 \in T \sim K^\circ$ such that $\|f(t_0)\| = 1$.

To the contrary, suppose there exists a compact set $K \subseteq T$, such that

$\sup_{t \in T \sim K^\circ} \|f(t)\| = 1$ and $\|f(t)\| < 1$, for all $t \in T \sim K^\circ$. Let $F_0(t) = \|f(t)\|$,

for all $t \in T$. Then F_0 is a bounded continuous function on T and thus has an extension, F , to βT , the Stone-Čech compactification of T . Since

$\sup_{t \in T \sim K^\circ} \|f(t)\| = 1$, we have $A = F^{-1}(1) \cap (\beta T \sim T) \neq \emptyset$. Also

$$A = \bigcap_{n=1}^{\infty} \{t \in T \sim K : F(t) > 1 - \frac{1}{n}\}$$

and thus A is a G_δ set in βT . By Čech [2], singletons in $\beta T \setminus T$ are not G_δ sets, so A contains at least two distinct points p and q .

Let $\{p_\mu\}$ and $\{q_\nu\}$ be disjoint nets contained in T such that $p_\mu \rightarrow p$ and $q_\nu \rightarrow q$ in βT . For each μ , let $\phi_\mu \in E^*$, with $\|\phi_\mu\| = 1$ and $\phi_\mu(f(p_\mu)) = F(p_\mu)$. Also, for each ν , choose $\psi_\nu \in E^*$, with $\|\psi_\nu\| = 1$ and $\psi_\nu(f(q_\nu)) = F(q_\nu)$. Let $\phi_\mu = \delta_{\phi_\mu, p_\mu}$ and $\psi_\nu = \delta_{\psi_\nu, q_\nu}$, for each μ and ν . Then ϕ_μ and $\psi_\nu \in C(T, E)^*$ and $\|\phi_\mu\| = \|\psi_\nu\| = 1$, for each μ and ν .

Since the ball in $C(T, E)^*$ is w^* -compact, there exist $\phi, \psi \in C(T, E)^*$, with $\|\phi\| \leq 1$ and $\|\psi\| \leq 1$, such that ϕ is a w^* -accumulation point of the net $\{\phi_\mu\}$ and ψ is a w^* -accumulation point of the net $\{\psi_\nu\}$. By construction, $\phi(f) = \psi(f) = 1$ and, thus, ϕ and ψ are support functionals to the ball in $C(T, E)$ at f . Since f is assumed to be a smooth point in $C(T, E)$, it must be that $\phi = \psi$. We will show that this is impossible. Let $P = \{p_\mu\} \cup \{p\}$ and $Q = \{q_\nu\} \cup \{q\}$. Then P and Q are disjoint closed subsets of βT , which is a compact Hausdorff space and therefore normal. Let $h_1, h_2 \in C(\beta T)$ with $0 \leq h_1, h_2 \leq 1$, $h_1 + h_2 = 1$ and $h_1(P) = h_2(Q) = 0$. Use h_i for the restriction of h_i to T , as well.

Clearly, if $g \in C(T, E)$, then $h_1 g \in \ker \phi$ and $h_2 g \in \ker \psi$. Since $\phi = \psi$, and $g \in C(T, E)$ can be written as $g = h_1 g + h_2 g$, we have $\phi = \psi = 0$. But, this contradicts $\|\phi\| = \|\psi\| = 1$. Therefore, we must have that either

$\sup_{t \in T \setminus K^0} \|f(t)\| < 1$, or there exists $t_0 \in T \setminus K^0$ with $\|f(t_0)\| = 1$, for any

compact set $K \subseteq T$.

(III) Finally we show that (i), (ii), (iii) hold. Taking $K = \emptyset$ in (II), since $\|f\| = 1$, we see that there exists a $t_0 \in T$ with $\|f(t_0)\| = 1$; and from (I), $\|f(t_0)\| > \|f(t)\|$ for all $t \neq t_0$. Again by (II), if $K \subseteq T$ is a compact set with $t_0 \in K^0$, then $\sup_{t \in T \setminus K} \|f(t)\| < 1$. If there exist distinct functionals $\phi_1, \phi_2 \in E^*$ with $\|\phi_1\| = \|\phi_2\| = 1$ such that $\phi_1(f(t_0)) = \phi_2(f(t_0)) = 1$, then this implies that $\delta_{\phi_1, t_0}, \delta_{\phi_2, t_0} \in C(T, E)^*$ are distinct support functionals to the ball in $C(T, E)$ at f , which contradicts the fact that f is a smooth point. Therefore $f(t_0)$ is a smooth point of E .

B.. Conversely, suppose that $f \in C(T, E)$, $\|f\| = 1$, and (i), (ii), and (iii) hold; then there exists a unique $t_0 \in T$ such that $\|f(t_0)\| = 1$, there exists a compact set $K \subseteq T$ with $t_0 \in K^\circ$ such that $\sup_{t \in T \sim K} \|f(t)\| < 1$, and E is smooth at $f(t_0)$.

Let $g \in C(T, E)$, $g \neq 0$, and let $\delta > 0$ be such that $\|f(t)\| < \|f(t_0)\| - \delta$ for all $t \in T \sim K$. If $0 < |\lambda| < \frac{\delta}{2\|g\|}$, then for $t \in T \sim K$ we have

$$\|f(t) + \lambda g(t)\| \leq \|f(t)\| + |\lambda| \|g(t)\| < \|f(t_0)\| + |\lambda| \|g\| - \delta < \|f(t_0)\| - \frac{\delta}{2}.$$

Thus, $\|f(t) + \lambda g(t)\| < \|f(t_0)\| - \frac{\delta}{2}$ for all $t \in T \sim K$ whenever $0 < |\lambda| < \frac{\delta}{2\|g\|}$.

On the other hand,

$$\|f + \lambda g\| \geq \|f\| - |\lambda| \|g\| = \|f(t_0)\| - |\lambda| \|g\| > \|f(t_0)\| - \frac{\delta}{2} \quad \text{for } 0 < |\lambda| < \frac{\delta}{2\|g\|}.$$

Therefore, for $0 < |\lambda| < \frac{\delta}{2\|g\|}$,

$$\sup_{t \in T} \|f(t) + \lambda g(t)\| = \sup_{t \in K} \|f(t) + \lambda g(t)\|.$$

Since K is compact, by Kondagunta's result [6],

$$\lim_{\lambda \rightarrow 0} \frac{\sup_{t \in K} \|f(t) + \lambda g(t)\| - \sup_{t \in K} \|f(t)\|}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda}$$

exists for all $g \in C(T, E)$. Hence $C(T, E)$ is smooth at f .

q.e.d.

An analogous result holds for the Fréchet differentiability of the norm in $C(T, E)$.

COROLLARY 3.2. Let T be a locally compact Hausdorff space and E a Banach space. The norm function $\|\cdot\| : C(T, E) \rightarrow \mathbb{R}^+$ is Fréchet differentiable at $f \in C(T, E)$, $f \neq 0$, if and only if

(i) there exists a unique $t_0 \in T$ such that $\|f(t_0)\| > \sup_{t \neq t_0} \|f(t)\|$;

(ii) $\{t_0\}$ is an open subset of T , that is, t_0 is an isolated point of T ;

(iii) the norm function $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$.

(Note: (ii) follows from (i).)

PROOF.

A. Suppose that $\|\cdot\| : C(T, E) \rightarrow \mathbb{R}^+$ is Fréchet differentiable at $f \in C(T, E)$, $\|f\| = 1$; then the ball in $C(T, E)$ is smooth at f , so there exists a $t_0 \in T$ and a compact neighborhood K of t_0 such that

$$(1) \|f(t_0)\| > \|f(t)\| \quad \text{for all } t \in T, \quad t \neq t_0,$$

$$(2) \sup_{t \in T \sim K} \|f(t)\| < 1,$$

$$(3) E \text{ is smooth at } f(t_0).$$

Now

$$\lim_{\lambda \rightarrow 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\sup_{t \in K} \|f(t) + \lambda g(t)\| - \sup_{t \in K} \|f(t)\|}{\lambda}$$

exists uniformly for $g \in B_{C(T, E)}$, and since K is compact, an appeal to the result of Cox and Nadler [3] shows that $\{t_0\}$ is open and $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$. Also, since $\{t_0\}$ is open and $K \sim \{t_0\}$ is compact, the uniqueness of t_0 shows that $\sup_{t \neq t_0} \|f(t)\| < 1$.

B. Conversely, suppose that (i), (ii), and (iii) hold. Using the previous theorem, we see that $\|\cdot\| : C(T, E) \rightarrow \mathbb{R}^+$ is Gateaux differentiable at f , and taking $K = \{t_0\}$,

$$\lim_{\lambda \rightarrow 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\|f(t_0) + \lambda g(t_0)\| - \|f(t_0)\|}{\lambda}$$

exists uniformly for $g \in B_{C(T, E)}$ since $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$.

q.e.d.

4. APPLICATIONS.

A. $\ell_\infty(E)$. If E is a Banach space, then

$$\ell_\infty(E) = \{x = \{x_n\}_{n \geq 1} \mid x_n \in E \text{ for } n \geq 1 \text{ and } \sup_{n \geq 1} \|x_n\| < \infty\}$$

with the supremum norm, $\|x\| = \sup_{n \geq 1} \|x_n\|$.

THEOREM 4.1. Let E be a Banach space, then the norm function $\|\cdot\| : \ell_\infty(E) \rightarrow \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $x = \{x_n\}_{n \geq 1}$, $x \neq 0$, if and only if

- (i) there exists an n_0 such that $\|x_{n_0}\| > \|x_n\|$ for $n \neq n_0$,
- (ii) $\sup_{n \neq n_0} \|x_n\| < \|x\|$,
- (iii) the norm function $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at x_{n_0} .

PROOF. Let $\mathbb{N}$ denote the set of positive integers equipped with the discrete topology, then $\ell_\infty(E) = C(\mathbb{N}, E)$ the space of bounded continuous E -valued functions on the locally compact Hausdorff space $\mathbb{N}$.

q.e.d.

B. $B(\ell_1, E)$. Let E be a Banach space, let ℓ_1 be the Banach space of all absolutely summable real valued sequences with $\|a\| = \sum_{n=1}^{\infty} |a_n|$ for $a = \{a_n\}_{n \geq 1} \in \ell_1$, and let $B(\ell_1, E)$ be the space of all bounded linear operators from ℓ_1 into E . For $n \geq 1$, let δ^n be the n^{th} basis vector in ℓ_1 , that is $\delta^n = \{\delta_k^n\}_{k \geq 1}$.

THEOREM 4.2. Let E be a Banach space, then the norm function $\|\cdot\| : B(\ell_1, E) \rightarrow \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $T \in B(\ell_1, E)$, $T \neq 0$, if and only if

- (i) there exists an n_0 such that $\|T(\delta^{n_0})\| > \|T(\delta^n)\|$ for $n \neq n_0$;
- (ii) $\sup_{n \neq n_0} \|T(\delta^n)\| < \|T\|$;
- (iii) the norm function $\|\cdot\| : E \rightarrow \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $T(\delta^{n_0})$.

PROOF. The mapping $\sigma : B(\ell_1, E) \rightarrow \ell_\infty(E)$ given by $\sigma(T) = \{T(\delta^n)\}_{n \geq 1}$ for $T \in B(\ell_1, E)$, is a linear isometry of $B(\ell_1, E)$ onto $\ell_\infty(E)$.

REMARKS.

1. In connection with the second example, it should be mentioned that Kheinrikh [5] has given a complete characterization of the points of Gateaux and Fréchet

differentiability of the norm in $K(E, F)$, the space of compact linear operators from E into F , where E and F are Banach spaces. He has also given a characterization of the points of Fréchet differentiability of the norm in $B(E, F)$, the space of bounded linear operators from E into F (no proofs are given in this paper). However, the more difficult question of smoothness in $B(E, F)$ is still unanswered.

2. Regarding Theorem 3.1. perhaps this will clear up the popular misconception that, for T locally compact Hausdorff, $C(T, E)$ (or $C(T, \mathbb{R})$) is smooth at f if and only if f peaks at some $t_0 \in T$. (See e.g. Holmes [4], p. 232, #4.10). A result which is obviously false, as the example in the introduction demonstrates.

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