ON A GENERALIZATION OF CLOSE-TO-CONVEXITY

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ABSTRACT. A class $T_k$ of analytic functions in the unit disc is defined in which the concept of close-to-convexity is generalized. A necessary condition for a function $f$ to belong to $T_k$, radius of convexity problem and a coefficient result are solved in this paper.

KEY WORDS AND PHRASES. Close-to-convex functions, functions of bounded boundary rotation, necessary condition, radius of convexity, generalized Koebe function.

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1. INTRODUCTION.

This paper is directed to mathematical specialists or non-specialists familiar with multivalent functions [1], and to close-to-convex functions [2].

Let $V_k$ be the class of functions of bounded boundary rotation and $K$ be the class of close-to-convex functions. We generalize the concept of close-to-convexity in the following direction.

**Definition.** Let $f$ with $f(z) = cz + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E = \{z:|z|<1\}$, $|c|=1$ and $f'(z) \neq 0$. Then $f \in T_k$, $k \geq 2$, if there exist a function $g \in V_k$ such that, for $z \in E$

$$\Re \left( \frac{f'(z)}{g'(z)} \right) > 0.$$  \hspace{1cm} (1.1)

It is clear that $T_2 = K$.

Using a method by Kaplan [2], we have

**Theorem 1.** Let $f \in T_k$. Then with $z = re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \left( \Re \left( \frac{(zf'(z))'}{f'(z)} \right) \right) d\theta > \frac{k^2}{2}.$$  \hspace{1cm} (1.2)
REMARK 1. From theorem 1, we can interpret some geometric meaning for the class $T_k$. For simplicity, let us suppose that the image domain is bounded by an analytic curve $C$. At a point on $C$, the outward drawn normal has an angle $\arg[e^{i\theta}f'(e^{i\theta})]$. Then from (1.2), it follows that the angle of the outward drawn normal turns back at most $\frac{k\pi}{2}$. This is a necessary condition for a function $f$ to belong to $T_k$. It will be interesting to see if this condition is also sufficient.

REMARK 2. Goodman [3] defines the class $K(\beta)$ of functions as follows.

Let $f$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E$ and $f'(z) \neq 0$. Then for $\beta > 0$, $f \in K(\beta)$, if for $z = r e^{i\theta}$ and $\theta_1 < \theta_2$,

$$\int_{\theta_1}^{\theta_2} \left[ \frac{\text{Re} \left( z f'(z) \right)}{f'(z)} \right] d\theta > -\beta \pi$$

We note that $T_k \subseteq K\left(\frac{k}{2}\right)$.

2. MAIN RESULTS

From remark 2 and results given in [3] for the class $K(\beta)$, we have at once

**THEOREM 2.** Let $f \in T_k$.

(i) Denote by $L(r,f)$ the length of the image of the circle $|z| = r$ under $f$ and by $A(r,f)$ the area of $f(|z|=r)$. Then for $0 \leq r < 1$,

(a) $L(r,f) \leq L(r,F_k)$,

(b) $A(r,f) \leq A(r,F_k)$,

where $F_k$ is defined by, for $z \in E$,

$$F_k(z) = \frac{1}{(k+2)} \left[ \left( \frac{1+k}{1-z} \right)^{\frac{b_k}{k+1}} - 1 \right]$$

$$= z + \sum_{n=2}^{\infty} \frac{A_n(k)}{n} z^n \quad \text{(2.1)}$$

and clearly $F_k \in T_k$.

(ii) $|a_n| \leq A_n(k)$, $n = 2, 3, \ldots \ldots \ldots \ldots$, $k \geq 2$

where $A_n(k)$ is defined by (2.1). This result is sharp for each $n \geq 2$.

(iii) For $z = r e^{i\theta}$, $0 \leq r < 1$,

$$\frac{(1-r)^{b_k}}{(1+r)^{b_k+2}} \leq |f'(z)| \leq \frac{(1+r)^{b_k}}{(1-r)^{b_k+2}}$$

These bounds are sharp, equality being attained for the function $F_k$ defined by (2.1).
We also need the following result.

**Lemma 1** [4]. Let \( g \in V_k \). Then there are two starlike functions \( s_1 \) and \( s_2 \) such that for \( z \in E \)

\[
g'(z) = \frac{(s_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}}
\]

**Theorem 3.** \( f \in T_k \) if and only if

\[
f'(z) = \frac{(k_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(k_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}} , \quad k_1, k_2 \in \mathbb{C}
\]

**Proof:** From definition 1, we have

\[
f'(z) = g'(z)h(z) , \quad g \in V_k \text{ and } \text{Re} h(z) > 0.
\]

Using lemma 1, we know that there are two starlike functions \( s_1 \) and \( s_2 \) such that for \( z \in E \)

\[
g'(z) = \frac{(s_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}}
\]

Thus

\[
f'(z) = \frac{(s_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}} h(z) = \frac{(s_1(z)h(z))/z)^{\frac{1}{2}k+\frac{1}{2}}}{((s_2(z)h(z))/z)^{\frac{1}{2}k-\frac{1}{2}}}
\]

\[
= \frac{(k_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(k_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}}
\]

where \( k_1 \) and \( k_2 \) are two suitable selected close-to-convex functions.

**Lemma 2.** Let \( H \) be analytic and be defined as

\[
H(z) = (zg'(z))', \quad g \in V_k \text{ and } H(z) = \left( \frac{1}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{1}{4} - \frac{1}{2} \right) h_2(z),
\]

\( \text{Re} h_i(z) > 0, i=1,2, \quad h_i(0) = 1 \)

Then

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |H(z)|^2 \, d\theta \leq \frac{1 + (k^2 - 1)r^2}{1 - r^2} \quad (z = re^{i\theta})
\]

and

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |H'(z)| \, d\theta \leq \frac{k}{1 - r^2}
\]

**Proof:** By the representation formula due to Paatero [5], we can write
\[ H(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \, du(t), \]

where \[ \int_{0}^{2\pi} \, du(t) = 2\pi, \text{ and } \int_{0}^{2\pi} |du(t)| \leq k. \]

Let \( H(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \)

Then

\[ c_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-int} \, du(t), \text{ and so for } n \geq 1, \]

\[ |c_n| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |du(t)| \leq k, \]

Thus

\[ \frac{1}{2\pi} \int_{0}^{2\pi} |H(z)|^2 \, d\theta \leq \sum_{n=0}^{\infty} |c_n|^2 \left( \frac{2n}{2} \right) \frac{2n}{2} \left( 1 + \frac{2n}{2} \right) = \frac{1 + (k^2 - 1)r^2}{1 - r^2}. \]

Also

\[ H'(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it}}{(1 - ze^{-it})^2} \, du(t), \]

Thus

\[ \frac{1}{2\pi} \left( \int_{0}^{2\pi} |H'(z)| \, d\theta \right)^2 \leq \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{(1 - re^{-it})^2} \, d\theta \frac{1}{|du(t)|} \leq \frac{1}{2\pi} \int_{0}^{2\pi} |du(t)| \leq \frac{k}{1 - r^2}. \]

**THEOREM 4:** Let \( f \in T_k \). Then for \( n \geq 1 \),

\[ \left| a_{n+1} \right| - \left| a_n \right| \leq c(k)n, \]

where \( c(k) \) is a constant and depends only on \( k \).

**PROOF:** Since \( f \in T_k \), we have for \( z \in E \),

\[ f'(z) = g'(z)h(z), \quad gcV_k \text{ and } \text{Re } h(z) > 0. \]

Set

\[ F(z) = zf'(z), \quad \text{with } \]

\[ H(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z), \quad \text{Re } h_1(z) > 0, i = 1, 2, h_1(0) = 1. \]

Thus, for \( \xi \in E \) and \( n \geq 1 \);
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\[ |(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z-\xi| |F(z)| d\theta, \]

and by using lemma 1 and (2.2), we obtain

\[ |(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{2\pi r^{n+1}} \left[ \int_0^{2\pi} \frac{|s_1(z)|}{|s_2(z)|} H(z)h(z) + zh'(z) \right] d\theta, \]

(2.3)

where \( s_1, s_2 \) are starlike functions.

It is well-known [1] that for starlike function \( s \in S \),

\[ \frac{r}{(1+r)^2} \leq |s(z)| \leq \frac{r}{(1-r)^2} \]  

(2.4)

Let \( 0 < r < 1 \). Then by a result of Golusin [6, p. 162], there exists a \( z_1 \) with \( |z_1| = r \) such that for all \( z, |z| = r, \)

\[ |z-z_1||s_1(z)| \leq \frac{2r^2}{1-r^2}. \]  

(2.5)

From (2.3)-(2.5), we have

\[ |(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{2\pi r^{n+1}} \left( \frac{2r^2}{1-r^2} \right) \left( \frac{r}{(1-r)^2} \right) \int_0^{2\pi} |H(z)h(z)+zh'(z)| d\theta \]  

(2.6)

Now as in [7], we have with \( z = re^{i\theta} \)

\[ \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \leq \frac{1+3r^2}{1-r^2} \]  

and

\[ \frac{1}{2\pi} \int_0^{2\pi} |zh'(z)| d\theta \leq \frac{2r}{1-r^2}, \]  

where \( \text{Re} h(z) > 0 \).  

(2.7)

Also

\[ \frac{1}{2\pi} \int_0^{2\pi} |H(z)h(z)+zh'(z)| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |H(z)h(z)| d\theta + \frac{1}{2\pi} \int_0^{2\pi} |zh'(z)| d\theta \]

\[ \leq \frac{(1+(k^2-1)r^2)^{1/2}(1+3r^2)^{1/2}}{1-r^2} + \frac{2r}{1-r^2} \]  

(2.8)

by using Schwarz's inequality, lemma 2 and (2.7).

Hence from (2.6) and (2.8), we have

\[ |(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{r^{n+1}} \frac{2^{3k}}{2^{3k+1}} \left[ (1+(k^2-1)r^2)^{1/2} \right] \frac{1}{(1-r)^{1/2}}, \]
and so choosing $|\zeta| = r = \left(\frac{n}{n+1}\right)^2$, we obtain for $n \geq 1$

$$n^2|a_{n+1} - a_n| \leq \left[\left(1 + (k^2 - 1)r^2\right)^{\frac{1}{2}} + 1\right] e^2 z^{\frac{k+2}{2}} \left(\frac{4}{3}\right)^{k+1} n^{\frac{1}{2}(k+1)}$$

Thus

$$||a_{n+1} - a_n|| \leq c(k)n^{\frac{1}{2}(k-1)}.$$

The function $F_k$ defined by (2.1) shows that the index $\left(\frac{k}{2} - 1\right)$ is best possible.

We now evaluate the radius of convexity for the class $T_k$.

**THEOREM 5:** Let $f \in T_k$. Then the radius $R$ of the circle which $f$ maps onto a convex domain is given by

$$R = \frac{1}{2}\left[(k+2) - \sqrt{(k^2+4k)}\right].$$

The function $F_k$ defined by (2.1) shows that this result is best possible. In particular, when $k = 2$, $R = 2\sqrt{3}$, which is well known. This result also follows from the remark in [3,p.23].

**PROOF:** By definition

$$zf'(z) = ag'(z)h(z) \quad gcV_k; \quad Re h(z) > 0.$$ 

Thus

$$\frac{(zf'(z))'}{f'(z)} = \frac{(ag'(z))'}{g'(z)} + \frac{zh'(z)}{h(z)}$$

and so

$$Re \frac{(zf'(z))'}{f'(z)} \geq Re \frac{(ag'(z))'}{g'(z)} = |zh'(z)|$$

For $gcV_k$, it is well known [9] that, for $z = re^{i\theta}, 0 < r < 1$,

$$Re \frac{(ag'(z))'}{g'(z)} \geq \frac{r^2 - kr + 1}{1-r^2}$$

Hence

$$Re \frac{(zf'(z))'}{f'(z)} \geq \frac{r^2 - kr + 1}{1-r^2} - \frac{2r}{1-r^2} = \frac{r^2 - (k+2)r + 1}{1-r^2}$$

This gives the required result.

**REMARKS 3.**

(i). We also note that the extremal function $F_k(z)$ defined by (2.1) is the same function as $F_{n}^{\beta}(z)$ defined by equation (2.6) in [3]. As A. W. Goodman has pointed out that this function is sometime referred to as the generalized Koebe function.
(ii). We conjecture that the class $T_k$ is a proper subclass of the class $K(\beta)$ as defined in [3], since in the definition of $T_k$, $g \in V_k$ and we know that $g \in V_k$, $2 \leq k \leq 4$, is convex in one direction and all the functions in one direction form a proper subclass of the class of close-to-convex functions.

(iii). It remains open whether $T_k$ is a linear invariant family.

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REFERENCES

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