A CHARACTERIZATION OF THE DESARGUESIAN PLANES OF ORDER $q^2$ BY $SL(2,q)$

D.A. FOULSER
Mathematics Department
University of Illinois at Chicago Circle
Chicago, Illinois 60680

N.L. JOHNSON
Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242

T.G. OSTROM
Department of Pure and Applied Mathematics
Washington State University
Pullman, Washington 99164

(Received January 25, 1982)

ABSTRACT. The main result is that if the translation complement of a translation plane of order $q^2$ contains a group isomorphic to $SL(2,q)$ and if the subgroups of order $q$ are elations (shears), then the plane is Desarguesian. This generalizes earlier work of Walker, who assumed that the kernel of the plane contained $GF(q)$.

KEY WORDS AND PHRASES. Translation planes, translation complements, elations.


THEOREM. Let $\pi$ be a translation plane of order $q^2$, where $q = p^r$ and $p$ is a prime. Let $G \cong SL(2,q)$ be a subgroup of the translation complement of $\pi$ whose elements of order $p$ are elations. Then $\pi$ is a Desarguesian plane.

This theorem is a special case required in the classification of all translation planes $\pi$ of order $q^2$ which admit a collineation group $G \cong SL(2,q)$ [1, 2]. That classification is a generalization of the work of Walker and Schaeffer [3, 4], who assume, in addition, that the kernel of $\pi$ contains $GF(q)$.

To begin the proof, let $W$ be a vector space of dimension $2r$ over $GF(p)$. Since
\pi is a 4r-dimensional vector space over GF(p), we may represent \pi as W \oplus W so that the points of \pi are vectors \((x,y)\), where \(x, y \in W\). The components of \pi (i.e., the lines containing \((0,0)\)) have the form \([0,y]: y \in W\) and \([(x,xA): x \in W]\) for various GF(p)—linear transformations \(A: W \to W\). We will denote the components by their defining equations \(x = 0\) and \(y = xA\), respectively. Next, note that each Sylow p-subgroup \(Q\) of \(G\) is abelian and hence all the elements \((\neq I)\) of \(Q\) have the same elation axis. Let \(S\) denote the set of all components of \(\pi\) and let \(N\) be the subset of elation axes; thus \(|S| = q^{2r} + 1\) and \(|N| = q + 1\).

**Lemma 1.** (Hering [5], Ostrom [6]). We may coordinatize \(\pi\) as above such that

\[ G = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A, B, C, D \in K; AD - BC = I \right\} \]

where \(K\) is a field of \(2r \times 2r\) matrices over GF(p) and \(K \cong GF(q)\). Further, the elation axes (that is, the elements of \(N\)) have the form \(y = xA (A \in K)\) and \(x = 0\).

**Lemma 2.** There is an element \(g \in G\) such that the following conditions are satisfied: (i) \(|g| | q + 1\); (ii) \(|g| \leq p^{t} - 1\) for \(t < 2r\); and (iii) \(g\) fixes a component of \(\pi\) which is not in the set \(N\).

**Proof.** The integer \(s\) is a p-primitive prime divisor of \(q^2 - 1\) if \(s\) is a prime, \(s \mid q^2 - 1\), and \(s \mid p^t - 1\) for \(0 < t < 2r\) (hence \(s \mid q + 1\)). \(q^2 - 1\) has a p-primitive prime divisor \(s\) unless \(q = 8\) or \(q = p\) and \(p + 1 = 2^s\) [7]. In the first case, let \(|g| = s\) so that \(g\) satisfies conditions (i) and (ii); then \(g\) also satisfies condition (iii) because \(|g|\) is a prime and \(g\) permutes the \((q - 1)\) components in \(S \setminus N\). If \(q = 8\), choose \(g\) such that \(|g| = 9\). Since \(|S \setminus N| = 56 \neq 0 \pmod 3\), \(g\) must fix one of the elements of \(S \setminus N\). Finally, if \(q = p\) and \(p + 1 = 2^s\), choose \(h\) of order 8 in \(G\) and let \(g = h^2\). Then \(g^2\) has order 2 in \(G = \text{SL}(2,K)\), so \(g^2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}\) fixes every component of \(\pi\). Hence, \(h\) has orbits of lengths 1, 2, and 4 in \(S\), and since \(4 \mid p(p - 1)\) then \(h\) has an orbit of length 1 or 2 on \(S \setminus N\). Therefore \(g = h^2\) fixes an element of \(S \setminus N\).

**Lemma 3.** Choose \(g \in G\) so that \(g\) satisfies the conditions of Lemma 2, and let \(L(g)\) be the ring of matrices generated by \(g\) over GF(p). Then \(L(g)\) is a field \(\cong GF(q^2)\) and \(L(g)\) contains the subfield

\[ \mathbb{K} = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in K \right\}. \]

**Proof.** \(g \in G \subset \text{GL}(2,K)\) by Lemma 1. As a 2 \times 2 matrix over \(K\), \(g\) has a minimum
polynomial \( f(x) \) over \( K \) of degree \( \leq 2 \). Since \( |g| = q(q-1) \), then the degree of \( f \) is 2 and \( f \) is irreducible over \( K \). Therefore, \( g \) and \( K \) generate a field \( U \cong \mathbb{GF}(q^2) \) which contains \( L(g) \) as a subfield. Since \( |g| = p^t - 1 \) (for \( t < 2r \)), then \( L(g) = U \) and \( L(g) \triangleright \mathbb{K} \).

**Lemma 4.** Let \( g \) of Lemma 2 fix the component \( y = xT \) of \( \mathbb{S} \setminus \mathbb{N} \). Then \( \mathbb{K}[T] \) is a field isomorphic to \( \mathbb{GF}(q^2) \).

**Proof.** \( L(g) \) and hence \( \mathbb{K} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in K \right\} \) fix the component \( y = xT \), and thus \( K \) centralizes \( T \). \( T \) and the elements of \( K \) are \( 2r \times 2r \) matrices which act on a vector space \( V = V(2r,p) \) of dimension \( 2r \) over \( \mathbb{GF}(p) \). \( K \) makes \( V \) into a 2-dimensional vector space and \( T \) acts as a \( K \)-linear transformation of \( V \). Hence, the minimum polynomial \( f(x) \) of \( T \) over \( K \) has degree \( \leq 2 \). If \( T \) has an eigenvalue \( A \) in \( K \), then the distinct components \( y = xT \) and \( y = xA \) of \( \mathbb{S} \) must intersect, which is impossible. Therefore, \( T \) is irreducible over \( K \) and \( K[I] \cong \mathbb{GF}(q^2) \).

We can now complete the proof of the Theorem. Let \( \mathbb{S}^* \) denote the Desarguesian affine plane of order \( q^2 \) coordinatized by the field \( L = K[I] \); i.e., the points of \( \mathbb{S}^* \) are \( \{(x,y) : x,y \in L\} \) and the components of \( \mathbb{S}^* \) are \( \{y = xC : C \in L\} \cup \{x = 0\} \). Clearly, \( \mathbb{GL}(2,L) \) acts as a collineation group of \( \mathbb{S}^* \). We superimpose \( \mathbb{S}^* \) on \( \mathbb{S} \) by identifying the points of \( \mathbb{S}^* \) and \( \mathbb{S} \). Since \( K \subseteq L \) and \( T \in L \), the components \( y = xA \) of \( \mathbb{N} \) and \( y = xT \) are components both of \( \mathbb{S}^* \) and of \( \mathbb{S} \). Since \( G = \mathbb{SL}(2,K) \subseteq \mathbb{GL}(2,L) \), then \( G \) acts both as a collineation group of \( \mathbb{S}^* \) and of \( \mathbb{S} \). Finally, recall that \( \mathbb{SL}(2,K) \) acts transitively on the \( q(q-1) \) components of \( \mathbb{S}^* \) outside of \( \mathbb{N} \) (for example, the stabilizer subgroup in \( \mathbb{SL}(2,K) \) of a component of \( \mathbb{S}^* \) outside \( \mathbb{N} \) has order \( q+1 \)). Therefore, the images of \( y = xT \) under \( G \) constitute \( q(q-1) \) components both of \( \mathbb{S}^* \) and of \( \mathbb{S} \); so \( \mathbb{S}^* = \mathbb{S} \) as required.

The research of all three authors was partially supported by the National Science Foundation.

**References**

1. FOULSER, D.A. and JOHNSON, N.L. "The Translation Planes of Order \( q^2 \) that Admit \( \mathbb{SL}(2,q) \) as a Collineation Group, I, Even Order," to appear in J. of Algebra.

2. FOULSER, D.A. and JOHNSON, N.L. "The Translation Planes of Order \( q^2 \) that Admit \( \mathbb{SL}(2,q) \) as a Collineation Group, II, Odd Order" to appear in J. of Geometry.


Submit your manuscripts at http://www.hindawi.com