INFINITE MATRICES AND ABSOLUTE ALMOST CONVERGENCE

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(Received January 26, 1981 and in revised form August 16, 1982)

ABSTRACT. In 1973, Stieglitz [5] introduced a notion of F_B -convergence which provided a wide generalization of the classical idea of almost convergence due to Lorentz [1]. The concept of strong almost convergence was introduced by Maddox [3] who later on generalized this concept analogous to Stieglitz's extension of almost convergence [4]. In the present paper we define absolute F_B -convergence which naturally emerges from the concept of F_B -convergence.

KEY WORDS AND PHRASES. Infinite matrices, almost convergence, strong almost convergence, F_B -convergence, absolute F_B -convergence.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 40C05, 40D05.

1. INTRODUCTION.

Let ℓ_∞ , c , and c_0 denote respectively the Banach spaces of bounded, convergent, and null sequences $x = (x_k)$ of complex numbers with norm $\|x\| = \sup_k |x_k|$, and let v be the space of sequences of bounded variation, that is,

$$v = \{x: \|x\| \equiv \sum_{k=0}^{\infty} |x_k - x_{k-1}| < +\infty, x_{-1} = 0\}.$$

Suppose that $B = (B_i)$ is a sequence of infinite complex matrices with $B_i = (b_{np}(i))$. Then $x \in \ell_\infty$ is said to be F_B -convergent [5], to the value $\text{Lim } Bx$, if

$$\lim_{n \rightarrow \infty} (B_i x)_n = \lim_{n \rightarrow \infty} \sum_{p=0}^{\infty} b_{np}(i) x_p = \text{Lim } Bx,$$

uniformly for $i = 0, 1, 2, \dots$.

The space F_B of F_B -convergent sequences depends on the fixed chosen sequence $B = (B_i)$. In case $B = B_0 = (I)$ (unit matrix), the space F_B is same as c and, for

$B = B_1 = (B_i^{(1)})$, it is same as the space f of almost convergent sequences [1], where $B_i^{(1)} = (b_{np}^{(1)}(i))$ with

$$b_{np}^{(1)}(i) = \begin{cases} \frac{1}{n+1}, & i \leq p \leq i+n \\ 0 & \text{elsewhere} \end{cases}$$

Maddox [4] generalized strong almost convergence by saying that $x_p \rightarrow s[F_B]$ if and only if

$$\sum_p b_{np}(i) |x_p - s| \rightarrow 0 \quad (n \rightarrow \infty, \text{ uniformly in } i) \quad (1.1)$$

assuming that the series in (1.1) converges for each n and i .

In particular, if $B = B_0$, the $[F_B] = c$; if $B = B_1$, then $[F_B] = [f]$, the space of strongly almost convergent sequences [3]. We shall write $e_k = (0, 0, \dots, 0, 1 \text{ (} k\text{th entry), } 0, \dots)$ and $e = (1, 1, 1, \dots)$.

Let s be the space of all complex sequences and

$$\begin{aligned} d_B &= \{x \in s: \lim_{n \rightarrow \infty} B_n x \text{ exists for each } i\} \\ F_B &= \{x \in (d_B \cap \ell_\infty): \lim_{n \rightarrow \infty} t_n(i, x) \text{ exists uniformly in } i, \\ &\quad \text{and the limit is independent of } i\}, \end{aligned}$$

where

$$t_n(i, x) = \begin{cases} \sum_{p=j}^{\infty} b_{np}(i) x_p, & (n \geq 1) \\ \sum_{p=0}^{\infty} \beta_{0p}(i) x_p, & (n = 0) \\ 0, & (n = -1) \end{cases}$$

and

$$\beta_{0p}(i) = \begin{cases} 1 & \text{if } p = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\emptyset_n(i, x) = t_n(i, x) - t_{n-1}(i, x).$$

Therefore, we have

$$\emptyset_n(i, x) = \begin{cases} \sum_{p=0}^{\infty} [b_{np}(i) - b_{n-1,p}(i)] x_p, & (n \geq 1) \\ \sum_{p=0}^{\infty} \beta_{0p}(i) x_p, & (n = 0) \end{cases} \quad (1.2)$$

DEFINITION. Let $B = (B_i)$ be a sequence of infinite matrices with $B_i = (b_{np}(i))$. A sequence $x \in \ell_\infty$ is said to be absolutely F_B -convergent if $\sum_{n=0}^{\infty} |\theta_n(i, x)|$ converges uniformly for $i \geq 0$, and $\lim_{n \rightarrow \infty} t_n(i, x)$ which must exist should take the same value for all i . We denote the space of absolute F_B -convergent sequences by $v(B)$.

2. THE MAIN RESULT.

In this note, we denote by $(v, v(B))$ the set of matrices which give new classes of absolute B -conservative matrices and absolute almost B -conservative matrices.

Let A be any infinite complex matrix for which the p th row-sum converges for a given x for all x in some class.

We have

$$A_p x = (Ax)_p = \sum_{k=0}^{\infty} a_{pk} x_k$$

and

$$(B_i x)_n = \sum_{p=0}^{\infty} b_{np}(i) x_p.$$

Therefore,

$$\begin{aligned} (B_i Ax)_n &= \sum_{p=0}^{\infty} b_{np}(i) A_p x \\ &= \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk} x_k, \end{aligned}$$

and, assuming the interchange of order of summation can be justified (see lemma), we get that

$$(B_i Ax)_n = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} b_{np}(i) a_{pk} x_k \quad (2.1)$$

Now, by (1.2) and (2.1), we have

$$\begin{aligned} \theta_n(i, Ax) &= t_n(i, Ax) - t_{n-1}(i, Ax) \\ &= \begin{cases} \sum_{p=0}^{\infty} [b_{np}(i) - b_{n-1,p}(i)] A_p x, & (n \geq 1), \\ \sum_{p=0}^{\infty} \beta_{0p}(i) A_p x, & (n = 0), \end{cases} \\ &= \sum_{k=0}^{\infty} g_{nk}(i) x_k, \end{aligned} \quad (2.2)$$

where

$$g_{nk}^{(i)} = \sum_{p=0}^{\infty} [b_{np}^{(i)} - b_{n-1,p}^{(i)}] a_{pk}, \quad (n \geq 1),$$

$$\sum_{p=0}^{\infty} \beta_{0p}^{(i)} a_{pk}, \quad (n = 0).$$

THEOREM. Let $B = (B_i)$ be a sequence of infinite matrices with

$$\sup_n \sum_{p=0}^{\infty} |b_{np}^{(i)}| < \infty, \quad \text{for each } i.$$

Let A be an infinite matrix. Then $A: v \rightarrow v(B)$ if and only if

- (i) $\sup_{p,k} \left| \sum_{\ell=k}^{\infty} a_{p\ell} \right| < \infty,$
- (ii) there is an N such that for $r, i = 0, 1, 2, \dots$

$$\sum_{n=N}^{\infty} \left| \sum_{k=0}^r g_{nk}^{(i)} \right| \leq K \quad (\text{constant}),$$
- (iii) $(a_{pk})_{p \geq 0} \in v(B)$ for each k , and
- (iv) $(\sum_{k=0}^{\infty} a_{pk})_{p \geq 0} \in v(B).$

Let $A \in (v, v(B))$. For each k , let a_{pk} be F_B -convergent with limit α_k . And let $\sum_{k=0}^{\infty} a_{pk}$ be F_B -convergent with limit α . (In each case, limit is taken for $p \geq 0$).

If $x = (x_k) \in v$, then

$$\lim_{n \rightarrow \infty} t_n(i, Ax) = \alpha \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} (x_k - \lim_{k \rightarrow \infty} x_k) \alpha_k.$$

We use the following lemma in the proof.

LEMMA. If either the necessity part or the sufficiency part of the theorem holds, then, for $x \in v$,

$$\sum_{p=0}^{\infty} b_{np}^{(i)} \sum_{k=0}^{\infty} a_{pk} x_k = \sum_{k=0}^{\infty} x_k \sum_{p=0}^{\infty} b_{np}^{(i)} a_{pk}.$$

PROOF. If either $A: v \rightarrow v(B)$ or the conditions (i)-(iv) of the theorem hold, then by partial summation, for $x \in v$,

$$\sum_{k=0}^{\infty} a_{pk} x_k = \sum_{k=0}^{\infty} d_{pk} (x_k - x_{k-1})$$

where $d_{pk} = \sum_{\ell=k}^{\infty} a_{p\ell}$. Since condition (i) holds, d_{pk} is bounded for all p, k . Thus

$$\begin{aligned} \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk} x_k &= \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} d_{pk} (x_k - x_{k-1}) \\ &= \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{p=0}^{\infty} b_{np}(i) d_{pk}, \end{aligned}$$

(where the inversion is justified by absolute convergence)

$$= \sum_{k=0}^{\infty} x_k \sum_{p=0}^{\infty} b_{np}(i) a_{pk}$$

since

$$\lim_{k \rightarrow \infty} x_k \sum_{p=0}^{\infty} b_{np}(i) d_{pk} = 0.$$

PROOF OF THEOREM. Necessity. Condition (i) follows from the fact that $A: v \rightarrow \ell_{\infty}$.

Since $e_k, e \in v$, necessity of (iii) and (iv) is obvious.

It is clear that, for fixed p and j ,

$$x \rightarrow \sum_{k=0}^j a_{pk} x_k$$

is a continuous linear functional on v . We are given that, for all $x \in v$, it tends to a limit as $j \rightarrow \infty$ (for fixed p) and hence, by the Banach-Steinhaus Theorem [2], this limit $A_p x$ is also a continuous linear functional on v .

We observe that, although $\sum_{n=0}^{\infty} |\phi_n(i, Ax)|$ is uniformly convergent in i , it needs not be uniformly bounded in i . For example, if $\phi_0(i, Ax) = i$ and $\phi_n(i, Ax) = 0$ ($n \geq 1$ and i), then $\sum_{n=0}^{\infty} |\phi_n(i, Ax)|$ is uniformly convergent in $i \geq 0$ but not uniformly bounded. Now, we can say that uniform convergence bears only on the behaviour of $\phi_n(i, Ax)$ for sufficiently large n . Thus, by definition, there is an m such that

$$q_{m,i}(x) = \sum_{n=m}^{\infty} |\phi_n(i, Ax)|.$$

For $m \geq 0$, $i \geq 0$, $q_{m,i}$ is a continuous seminorm on v , and there is an integer N such that $\{q_{N,i}\}_{i \geq 0}$ is pointwise bounded on v . Such an N exists. For suppose not. Then for $r = 0, 1, 2, \dots$ there exists $x_r \in v$ with

$$\sup_{i \geq 0} q_{r,i}(x_r) = \infty.$$

By the principle of condensation of singularities [6],

$$\{x \in v: \sup_{i \geq 0} q_{r,i}(x) = \infty \text{ for } r = 0, 1, 2, \dots\}$$

is of second category in v and hence nonempty, i.e., there is $x \in v$ with

$$\sup_{i \geq 0} q_{r,i}(x) = \infty \quad \text{for } r = 0, 1, 2, \dots$$

But this contradicts the fact that to each $x \in v$ there exists an integer N_x with

$$\sup_{i \geq 0} q_{N_x,i}(x) < \infty.$$

Now, by another application of the Banach-Steinhaus Theorem, there exists a constant M such that

$$q_{N,i}(x) \leq M \|x\|. \quad (2.3)$$

Apply (2.3) with $x = (x_k)$ defined by $x_k = 1$ for $k \leq r$ and 0 for $k > r$. Hence (ii) must hold.

Sufficiency. Suppose that the conditions (i)-(iv) hold and that $x \in v$. We have defined $v(B)$ as a subspace of ℓ_∞ . Thus, in order to show that $Ax \in v(B)$, it is necessary to prove that Ax is bounded. By virtue of condition (i), this follows immediately.

Now, it follows from (iv) and the lemma that

$$\sum_{k=0}^{\infty} g_{nk}(i)$$

converges for all i, n . Hence, if we write

$$h_{nk}(i) = \sum_{\ell=k}^{\infty} g_{n\ell}(i),$$

then $h_{nk}(i)$ is defined, also for fixed i, n ,

$$h_{nk}(i) \rightarrow 0 \quad (2.4)$$

as $k \rightarrow \infty$. Now condition (iv) gives us that

$$\sum_{n=0}^{\infty} |h_{n0}(i)| \quad (2.5)$$

converges uniformly in i , and, for suitable chosen N ,

$$\sum_{n=N}^{\infty} |h_{n0}(i)| \quad (2.6)$$

is bounded. By virtue of condition (iii), for fixed k , we get that

$$\sum_{n=0}^{\infty} |g_{nk}(i)|$$

converges uniformly in i . Since

$$h_{nk}(i) = h_{n0}(i) - \sum_{\ell=0}^{k-1} g_{n\ell}(i), \quad (2.7)$$

it follows that, for fixed k ,

$$\sum_{n=0}^{\infty} |h_{nk}(i)| \quad (2.8)$$

converges uniformly in i .

Now

$$\begin{aligned} \emptyset(i, Ax) &= \sum_{k=0}^{\infty} g_{nk}(i) x_k \\ &= \sum_{k=0}^{\infty} [h_{nk}(i) - h_{n,k+1}(i)] x_k \\ &= \sum_{k=0}^{\infty} h_{nk}(i) (x_k - x_{k-1}), \end{aligned} \quad (2.9)$$

by (2.4) and the boundedness of x_k .

Condition (ii) and the boundedness of (2.6) show that

$$\sum_{n=N}^{\infty} |h_{nk}(i)| \quad (2.10)$$

is bounded for all k, i . We can make

$$\sum_{k=k_0+1}^{\infty} |x_k - x_{k-1}|$$

arbitrarily small by choosing k_0 sufficiently large. It therefore follows that, given

$\varepsilon > 0$, we can choose k_0 so that, for all i ,

$$\sum_{n=N}^{\infty} \left| \sum_{k=k_0+1}^{\infty} h_{nk}(i) (x_k - x_{k-1}) \right| < \varepsilon. \quad (2.11)$$

By the uniform convergence of (2.8), it follows that, once k_0 has been chosen, we can choose n_0 so that, for all i ,

$$\sum_{n=n_0+1}^{\infty} \left| \sum_{k=0}^{k_0} h_{nk}(i) (x_k - x_{k-1}) \right| < \varepsilon.$$

It follows from (2.11) that the same inequality holds when $\sum_{n=N}^{\infty}$ is replaced by $\sum_{n=n_0+1}^{\infty}$; hence, for all i ,

$$\sum_{n=n_0+1}^{\infty} \left| \sum_{k=0}^{\infty} h_{nk}(i) (x_k - x_{k-1}) \right| < 2\varepsilon. \quad (2.12)$$

Hence,

$$\sum_{n=n_0+1}^{\infty} |\emptyset_n(i, Ax)| < 2\varepsilon.$$

Thus

$$\sum_{n=0}^{\infty} |\emptyset_n(i, Ax)|$$

converges uniformly.

Now, by virtue of (2.9), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} t_n(i, Ax) - t_{N-1}(i, Ax) &= \sum_{n=N}^{\infty} \sum_{k=0}^{\infty} h_{nk}(i) (x_k - x_{k-1}) \\ &= \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{n=N}^{\infty} h_{nk}(i) \end{aligned} \quad (2.13)$$

the assertion being justified by absolute convergence because of the boundedness of (2.10). By (2.7), we have

$$\begin{aligned} \sum_{n=N}^{\infty} h_{nk}(i) &= \sum_{n=N}^{\infty} h_{n0}(i) - \sum_{\ell=0}^{k-1} \sum_{n=N}^{\infty} g_{n\ell}(i) \\ &= \alpha - \sum_{\ell=0}^{k-1} \alpha_{\ell} - \sum_{\ell=k}^{\infty} \sum_{p=0}^{\infty} b_{N-1,p}(i) a_{p\ell}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} t_n(i, Ax) = \alpha \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} (x_k - \lim_{k \rightarrow \infty} x_k) \alpha_k.$$

This completes the proof.

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