A NOTE ON AN EXTENSION OF LINDELÖF'S THEOREM TO MEROMORPHIC FUNCTIONS

MOHAMMAD SALMASSI
Department of Mathematics
University of Rhode Island
Kingston, Rhode Island 02881 U.S.A.

(Received April 27,1983)

ABSTRACT. S. M. Shah [3] has given an extension of Lindelöf's Theorem to meromorphic functions. He also obtained an expression for the characteristic function of a meromorphic function of integer order. In this note we give estimates for \( \log |f(re^{i\theta})| \) of such functions.

KEY WORDS AND PHRASES. meromorphic functions, proximate order, slowly changing functions.

AMS (MOS) SUBJECT CLASSIFICATION (1980) CODES. 30D30

1. INTRODUCTION.

In [3;theorem 1] S. M. Shah obtained an expression for the characteristic function \( T(r,f) \) of a meromorphic function \( f(z) \) of integer order \( \rho \). Following the argument of Cartwright [2; theorem 45,46] we can obtain the following results for \( \log |f(re^{i\theta})| \).

We write

\[
    n(r) = n(r,1/f) + n(r,f); \quad N(r) = N(r,1/f)+N(r,f).
\]

Since \( \rho \) is a positive integer, we can write \( f(z) \) in the form (see [3])

\[
    f(z) = z^k \exp(cz^\rho + \ldots) \prod_{l=1}^{\infty} E(z/a_n,\rho) \prod_{l=1}^{\infty} E(z/b_n,\rho). \quad (1.1)
\]

Let \( \rho(r) \) be a proximate order [3] for \( N(r) \) and let \( n_L = \limsup_{r \to \infty} n(r)/r^\rho L(r) \), where \( L(r) \) is a slowly changing function.

2. MAIN RESULTS.

THEOREM. Let \( f(z) \) be a meromorphic function of integer order \( \rho > 0 \) and let

\[
    S(r) = c + \frac{1}{\rho} \sum_{a_n} \frac{a^{-\rho}}{a_n} - \frac{1}{\rho} \sum_{b_n} \frac{b^{-\rho}}{b_n},
\]
i. Suppose \( n_L < \infty \). Then for every \( \eta > 0 \), there is a \( K(\rho, \eta) \) such that for every \( \epsilon > 0 \),
\[
\left| \log |f(re^{i\theta})| - \Re(r^\rho e^{i\theta} S(r)) \right| < K(\rho, \eta)(n_L + \epsilon)r^\rho L(r)
\] (2.1)
for \( 0 \leq r \leq R \), except perhaps in circles the sum of whose radii is less than \( n_R \), provided that \( R > R^o(\epsilon, \eta) \).

ii. Suppose \( N(r) \) is of order \( \rho \). Then there is a \( K(\rho, \eta) \) such that
\[
\left| \log |f(re^{i\theta})| - \Re(r^\rho e^{i\theta} S(r)) \right| < K(\rho, \eta)r^\rho (r)
\] (2.2)
for \( 0 < r < R \), except perhaps in circles the sum of whose radii is less than \( n_R \), provided that \( R > R^o(\eta) \).

iii. Let \( \limsup_{r \to R} \log N(r)/\log r = c_1 < \rho \) and let \( c_1 < c_2 < \rho \leq 1 + c_2 \). Then for every \( \eta > 0 \), there is a \( K(c_2, \eta) \) such that
\[
\left| \log |f(re^{i\theta})| - \Re(r^\rho e^{i\theta} S(r)) \right| < K(c_2, \eta)r^{c_2}
\]
for \( 0 \leq r \leq R \), except perhaps in circles the sum of whose radii is less than \( n_R \), provided that \( R > R^o(c_2, \eta) \). The proof depends on the following lemma of Cartan (see [1; p. 46], also [2; pp. 73-77]):

**Lemma (H. Cartan).** Let \( p(z) = \prod_{k=1}^{n} (z-z_k) \); for any positive \( H \), the inequality
\[
|p(z)| > (H/e)^n
\]
holds outside at most \( n \) circles the sum of whose radii is at most \( 2H \).

We omit the details of the proof of the theorem.

**REFERENCES**

Submit your manuscripts at http://www.hindawi.com