DOT PRODUCT REARRANGEMENTS

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ABSTRACT. Let $a = (a_n)$, $x = (x_n)$ denote nonnegative sequences; $x = (x_{\pi(n)})$ denotes the rearranged sequence determined by the permutation π , $a \cdot x$ denotes the dot product $\sum a_n x_n$; and $S(a, x)$ denotes $\{a \cdot x_\pi : \pi \text{ is a permutation of the positive integers}\}$. We examine $S(a, x)$ as a subset of the nonnegative real line in certain special circumstances. The main result is that if $a_n \uparrow \infty$, then $S(a, x) = [a \cdot x, \infty]$ for every $x_n \not\equiv 0$ if and only if a_{n+1}/a_n is uniformly bounded.

KEY WORDS AND PHRASES. Dot product, series rearrangements, conditional convergence.

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An elementary classical result of Riemann on infinite series states that a conditionally convergent series that is not absolutely convergent can be rearranged to sum to any extended real number. A slightly similar group of questions arose in connection with certain formulas in operator theory [1, p. 181]. Namely, if we let $a = (a_n)$, $x = (x_n)$ denote any two non-negative sequences and x_π denote the sequence $(x_{\pi(n)})$ where π is any permutation of the positive integers, then what can be said about the set of non-negative real numbers $S(a, x) = \{a \cdot x_\pi : \pi \text{ is a permutation of the positive integers}\}$. More specifically, which subsets of the non-negative real line can be realized as the form $S(a, x)$ for some such a and x ?

Various facts about $S(a, x)$ are obvious

- (1) $S(a, x) \subset [0, \infty]$. The values 0 and ∞ may be obtained.
- (2) If a and x are strictly positive sequences or are at most finitely zero, then $S(a, x) \subset (0, \infty]$.
- (3) Not all subsets of $[0, \infty]$ are realizable as an $S(a, x)$ set. This follows by a cardinality argument. If c denotes the cardinality of $[0, \infty]$, then the cardinality of the class of subsets of $[0, \infty]$ is 2^c , but the cardinality of the class of sequences a and x is c and thus the cardinality of the subsets $S(a, x)$ is less than or equal to $c \cdot c = c$.
- (4) If either a or x is finitely non-zero then $S(a, x)$ is countable.
- (5) An example: if $a = (0, 2, 0, 2, \dots)$ and $x = (3^{-n})$, then $S(a, x)$ is precisely the Cantor set except for those non-negative real numbers whose ternary expansion consists of a tail of 0's or a tail of 2's (i.e., a subset of the rational numbers.),

It seems too ambitious to consider the general question at this time. For this reason we shall restrict our attention to the cases when a is a non-decreasing sequence and x is a non-increasing sequence,

If $a \equiv 0$ or $x \equiv 0$, the problem is trivial and $S(a, x) = \{0\}$. If $a_1 \neq 0$ and $x_n \neq 0$, the problem is trivial and $S(a, x) = \{\infty\}$. If a_n is bounded by M , then $S(a, x) \subset [0, M \sum x_n]$. In any case, hereafter we shall assume $a_n \uparrow \infty$ and $x_n \downarrow 0$, unless otherwise specified.

The Lemma that follows is a well-known fact, but we give a proof for completeness and because the proof contains some of the ideas used in the main result.

LEMMA. If $a_n \uparrow$ and $x_n \downarrow$ then $S(a, x) \subset [a \cdot x, \infty]$. In addition, $a \cdot x \in S(a, x)$, and if $a_n \uparrow \infty$ and $x_n \neq 0$ for all n or if $a_n \uparrow$ and $a_n > 0$ for some n and $x_n \neq 0$, then $\infty \in S(a, x)$.

PROOF. It suffices to show that for every permutation π of the positive integers, we have $a \cdot x \leq \sum a_n x_{\pi(n)}$ or, equivalently, $a \cdot x \leq \sum a_{\pi(n)} x_n$ for every π . The rest of the lemma is clear.

Define π_1 in terms of π as follows. Set

$$\pi_1(n) = \begin{cases} 1 & n = 1 \\ \pi(1) & n = \pi^{-1}(1) \\ \pi(n) & \text{otherwise} \end{cases}$$

It is straightforward to verify that π_1 is also a permutation of the positive integers (one-to-one and onto) which fixes 1. We assert that $a_{\pi_1} \cdot x \leq a_{\pi} \cdot x$. To see this, note that $\pi(1) \geq 1$ and $\pi^{-1}(1) \geq 1$. Hence $a_{\pi(1)} - a_1 \geq 0$ and $x_1 - x_{\pi^{-1}(1)} \geq 0$. Therefore

$$\begin{aligned} \sum (a_{\pi(n)} - a_{\pi_1(n)})x_n &= (a_{\pi(1)} - a_{\pi_1(1)})x_1 + (a_{\pi(\pi^{-1}(1))} - a_{\pi_1(\pi^{-1}(1))})x_{\pi^{-1}(1)} \\ &= (a_{\pi(1)} - a_1)(x_1 - x_{\pi^{-1}(1)}) \\ &\geq 0. \end{aligned}$$

Proceeding inductively, we obtain a sequence of permutations π_k that fix $1, 2, \dots, k$ for which $a_{\pi_k} \cdot x \leq a_{\pi_{k-1}} \cdot x$. Hence, for every k ,

$$\sum_{n=1}^k a_n x_n = \sum_{n=1}^k a_{\pi_k(n)} x_n \leq a_{\pi_k} \cdot x \leq a_{\pi} \cdot x.$$

Letting $k \rightarrow \infty$, we obtain $a \cdot x \leq a_{\pi} \cdot x$.

The main question of this paper is: for which a, x with $a_n \uparrow \infty$ and $x_n \downarrow 0$ is $S(a, x) = [a \cdot x, \infty]$?

The main result of this paper gives a partial answer. Namely, we can characterize which $a_n \uparrow \infty$ have the property that $S(a, x) = [a \cdot x, \infty]$ for every x such that $x_n \downarrow 0$.

On first sight, it might appear that $S(a, x)$ can never be $[a \cdot x, \infty]$ or that it is quite rare. The first result in this direction was that if $a_n = n$ for every n , then $S(a, x) = [a \cdot x, \infty]$ for every x such that $x_n \neq 0$. That $S(a, x)$ may not be $[a \cdot x, \infty]$ was first decided by an example due to Robert Young. Namely, let $a_n = 2^{2^n}$ and $x_n = 2^{-2^{n+1}}$. Both results are unpublished. The succeeding results and techniques are due to the work of the authors in collaboration with Hugh Montgomery.

THEOREM 1. (The Main Theorem) Let $a = (a_n)$ where $a_n > 0$ for every n and $a_n \rightarrow \infty$. Consider the following conditions:

- (1) a_{n+1}/a_n is bounded.
- (2) For the non-negative sequence $x = (x_n)$, there exist subsequences (a_{n_k}) and (x_{m_k}) of a and x respectively such that
 - (a) $a_{n_k} x_{m_k} \rightarrow 0$ as $k \rightarrow \infty$, and
 - (b) $\sum_k a_{n_k} x_{m_k} = \infty$.
- (3) $S(a, x) = [a \cdot x, \infty]$.

Then (1) implies (2) for every strictly positive sequence $x = (x_n)$ that tends to 0. Also if $a_n \uparrow \infty$ and $x_n \downarrow 0$ where $a_n, x_n \neq 0$ for all n , then (2) implies (3).

PROOF. To prove that (1) implies that (2) holds for every strictly positive sequence $x = (x_n)$ that tends to 0, suppose $a_{n+1}/a_n \leq M$ for all n . We assert that for every positive integer k , there exist arbitrarily large positive integers n_k and m_k for which $(k+1)^{-1} \leq a_{n_k} x_{m_k} \leq M k^{-1}$. If this assertion were true, then clearly we could choose two strictly increasing subsequences of positive integers (n_k) and (m_k) such that $a_{n_k} x_{m_k} \rightarrow 0$ as $k \rightarrow \infty$ to prove the assertion.

For each fixed positive integer k , $(k+1)^{-1} \leq a_n x_m \leq M k^{-1}$ if and only if $x_m \in [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$. All we need show is that there exist arbitrarily large n, m for which $x_m \in [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$.

Suppose to the contrary that there exists a positive integer N for which $x_m \notin [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$ for every $n, m \geq N$. In other words, for every $m \geq N$, $x_m \notin \bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$. (Note: This would imply that $\bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}]$ cannot contain any interval of the form $(0, \epsilon)$ for some $\epsilon > 0$, since $x_m \rightarrow 0$ as $m \rightarrow \infty$. However, this is not the case. Indeed, the proof below can be used to show that for every N , there exists $\epsilon > 0$ such that

$$(0, \epsilon) \subset \bigcup_{n \geq N} [(a_n(k+1))^{-1}, M(a_n k)^{-1}].$$

For each $m \geq N$, let n_m denote the least positive integer n such that $M(a_{n+1} k)^{-1} < x_m$, which exists since $a_n \rightarrow \infty$ as $n \rightarrow \infty$ and hence $M(a_{n+1} k)^{-1} \rightarrow 0$

as $n \rightarrow \infty$. For m sufficiently large, we have $M(a_{n_m+1} k)^{-1} \leq x_m \leq M(a_{n_m} k)^{-1}$.

Also, since $M(a_{n_m+1} k)^{-1} < x_m$ and $x_m \rightarrow 0$ as $m \rightarrow \infty$, we have $m \rightarrow \infty$ implies

$a_{n_m+1} \rightarrow \infty$ and hence $n_m \rightarrow \infty$. Therefore $n_m \geq N$ for all m sufficiently large,

and for these m , $x_m \notin [(a_{n_m} (k+1))^{-1}, M(a_{n_m} k)^{-1}]$. Hence, for infinitely many m ,

we have $x_m \leq M(a_{n_m} k)^{-1}$ and $x_m \notin [(a_{n_m} (k+1))^{-1}, M(a_{n_m} k)^{-1}]$. Therefore, for

infinitely many m , we have $M(a_{n_m+1} k)^{-1} < x_m < (a_{n_m} (k+1))^{-1}$. This implies that

$M(a_{n_m+1} k)^{-1} < (a_{n_m} (k+1))^{-1}$ for infinitely many m , or equivalently,

$a_{n_m+1}/a_{n_m} > M(k+1)/k > M$ for infinitely many m , which contradicts our assumption

that $a_{n+1}/a_n \leq M$ for all n . Hence (2) is proved.

To prove (2) $\rightarrow$ (3) whenever $a_n \uparrow \infty$ and $x_n \downarrow 0$, suppose (2) holds for a and x , so that there exist subsequences (a_{n_k}) and (x_{m_k}) such $a_{n_k} x_{m_k} \rightarrow 0$ as $k \rightarrow \infty$, and $\sum_k a_{n_k} x_{m_k} = \infty$. We first assert that without loss of generality we may assume that $a \cdot x = \sum_n a_n x_n < \infty$. To see this suppose $a \cdot x = \sum_n a_n x_n = \infty$. Then by the lemma we have that $S(a, x) = \{\infty\}$, and hence (3) holds.

Assuming that $\sum_n a_n x_n < \infty$, we next assert that without loss of generality we can assume that $n_k > m_k$ for every k . To see this, let Z_1 denote the set $\{k : n_k > m_k\}$ and let Z_2 denote the set $\{k : n_k \leq m_k\}$. Then

$$\infty = \sum_k a_{n_k} x_{m_k} = \sum_{k \in Z_1} a_{n_k} x_{m_k} + \sum_{k \in Z_2} a_{n_k} x_{m_k}$$

But $\sum_{k \in Z_2} a_{n_k} x_{m_k} \leq \sum_{k \in Z_2} a_{n_k} x_{n_k} \leq \sum_n a_n x_n < \infty$. Therefore $\sum_{k \in Z_1} a_{n_k} x_{m_k} = \infty$. Let Z_1

determine subsequences of (n_k) and (m_k) , which for simplicity we again call (n_k) and (m_k) , respectively, by taking only those entries n_k, m_k (in increasing order) for which $k \in Z_1$. This gives us subsequences (a_{n_k}) and (x_{m_k}) of a and x which satisfy conditions a and b in the 2nd condition of the theorem, and in addition satisfy $n_k > m_k$ for all k .

Next we assert that without loss of generality we may assume $n_k \neq m_j$ for all k, j . To see this, note that we have $n_k > m_k$ for all k and that $\langle n_k \rangle$ and $\langle m_k \rangle$ are strictly increasing (a property of subsequences). Therefore if $n_k = m_j$ for

some k, j , then $k < j$ and $n_k \neq m_j$ for all $i \neq j$. That is, n_k can occur at most once among the m_j 's. Put $(n_1, m_1), \dots, (n_{k_1}, m_{k_1}) \in S_1$ where k_1+1 is the least positive integer such that $m_{k_1+1} = n_k$ for some $k < k_1 + 1$. Put $(n_{k_1+1}, m_{k_1+1}), \dots, (n_{k_2}, m_{k_2}) \in S_2$ where k_2+1 is the least positive integer, if it exists, such that $m_{k_2+1} = n_k$ for some $k_1+1 \leq k < k_2+1$. Put $(n_{k_2+1}, m_{k_2+1}), \dots, (n_{k_3}, m_{k_3}) \in S_1$ such that k_3+1 is the least positive integer, if it exists, such that $m_{k_3+1} = n_k$ for some $k \leq k_1$ or $k_2 \leq k < k_3+1$. Continuing in this way, if no such least positive integer exists, then either S_1 or S_2 is finite. Otherwise both S_1, S_2 are infinite. For either case, no $n_k = m_j$ when both $(n_k, m_k), (n_j, m_j) \in S_1$ or S_2 . Then clearly S_1, S_2 is a disjoint partition of the set of all (n_k, m_k) and in each set, no n_k appears as an m_j . Therefore $\infty = \sum_{n_k} x_{m_k} = \sum_{S_1} a_{n_k} x_{m_k} + \sum_{S_2} a_{n_k} x_{m_k}$, and so either $\sum_{S_1} a_{n_k} x_{m_k} = \infty$ or $\sum_{S_2} a_{n_k} x_{m_k} = \infty$. Choosing S_1 or S_2 accordingly we produce the sequence (n_k, m_k) with the desired properties, (i.e., satisfying a) and b) in Theorem 1 and also satisfying $n_k \neq m_j$ for all k, j and $n_k > m_k$ for every k).

Now consider the series $\sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$. Since $n_k > m_k$, we have $0 \leq a_{n_k} - a_{m_k} \leq a_{n_k}$ and $0 \leq x_{m_k} - x_{n_k} \leq x_{m_k}$, and so $0 \leq (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) \leq a_{n_k} x_{m_k} \rightarrow 0$ as $k \rightarrow \infty$. Furthermore, since $\sum_k a_{n_k} x_{m_k} = \infty$, $a_{m_k} x_{n_k} \geq 0$, $\sum_k a_{n_k} x_{n_k} \leq a \cdot x < \infty$, $\sum_k a_{m_k} x_{m_k} \leq a \cdot x < \infty$, and $\sum_k a_{m_k} x_{n_k} \leq \sum_k a_{n_k} x_{n_k} < \infty$, we have

$$\begin{aligned} \sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) &= \sum_k (a_{n_k} x_{m_k} + a_{m_k} x_{n_k} - a_{n_k} x_{n_k} - a_{m_k} x_{m_k}) \\ &= \infty. \end{aligned}$$

We shall now show that for every $\varepsilon > 0$, there exists a subsequence (k_n) of positive integers such that $\varepsilon = \sum_{k \in \{k_n\}} (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$. This follows from the following more general fact.

Suppose $(d(k))$ is a non-negative sequence for which $d(k) \rightarrow 0$ as $k \rightarrow \infty$ and $\sum d(k) = \infty$. We assert that very every $\varepsilon > 0$, there exists a subsequence (k_n) such that $\varepsilon = \sum d(k_n)$. The proof of this fact proceeds along the same lines as the proof of Riemann's theorem on rearrangements of conditionally convergent series. Fix

$\varepsilon > 0$ and choose $n_1 \geq N_1$ so that $d(k) < \varepsilon$ for every $k \geq N_1$, and so that n_1 is the greatest integer greater than N_1 such that $\sum_{k=N_1}^{n_1} d(k) < \varepsilon$. Hence $\sum_{k=N_1}^{n_1+1} d(k) < \varepsilon \leq \sum_{k=N_1}^{n_1} d(k)$. This can be done since $d(k) \rightarrow 0$ as $k \rightarrow \infty$ and $d(k) = \infty$. Choose $N_2 > n_1$ so that $d(k) < (\varepsilon - \sum_{k=N_1}^{n_1} d(k))/2$ for every $k \geq N_2$ and then choose n_2 to be the largest integer greater than N_2 such that $\sum_{k=N_2}^{n_2} d(k) < \varepsilon - \sum_{k=N_1}^{n_1} d(k)$. Hence $\sum_{k=N_2}^{n_2+1} d(k) < \varepsilon - \sum_{k=N_1}^{n_1} d(k) \leq \sum_{k=N_2}^{n_2} d(k)$. Proceeding inductively in this way, we obtain sequences (N_p) and (n_p) of positive integers for which $n_p \geq N_p > n_{p-1}$, $0 \leq d(k) \leq (\varepsilon - \sum_{q=1}^{p-1} \sum_{k=N_q}^{n_q} d(k))/2^{p-1}$ for every p and every $k \geq N_p$, and

$$\sum_{k=N_p}^{n_p} d(k) < \varepsilon - \sum_{q=1}^{p-1} \sum_{k=N_q}^{n_q} d(k) \leq \sum_{k=N_p}^{n_p+1} d(k).$$

This implies that

$$\begin{aligned} 0 < \varepsilon - \sum_{q=1}^p \sum_{k=N_q}^{n_q} d(k) &\leq d(n_p + 1) \leq (\varepsilon - \sum_{q=1}^{p-1} \sum_{k=N_q}^{n_q} d(k))/2^{p-1} \\ &\leq \varepsilon/2^{p-1} \rightarrow 0 \text{ as } p \rightarrow \infty. \end{aligned}$$

Therefore $\varepsilon = \sum_{q=1}^{\infty} \sum_{k=N_q}^{n_q} d(k)$. Hence, if we choose (k_n) to be the strictly increasing sequence of positive integers k , where k is taken to range over the set $\bigcup_{p=1}^{\infty} \{k : N_p \leq k \leq n_p\}$, we have $\varepsilon = \sum d(k_n)$.

Applying this result to the sequence $(a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$, since it is non-negative, tends to 0, and sums to ∞ , we obtain that for every $\varepsilon > 0$, there exist subsequences of (n_k) and (m_k) , which we shall again denote by (n_k) and (m_k) , for which $\varepsilon = \sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k})$.

Now recall that we wish to show that $S(a, x) = [a \cdot x, \infty]$. We already know $a \cdot x$ and $\infty \in S(a, x)$. Suppose $a \cdot x < r < \infty$. It suffices to show $r \in S(a, x)$. Let $\varepsilon = r - a \cdot x$ and choose subsequences which we again call (n_k) and (m_k) so that

$$\varepsilon = \sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}).$$

We now choose π , the requisite permutation on Z^+ , as follows. Let $\pi(n_k) = m_k$ and $\pi(m_k) = n_k$ for each k , and let π fix all other integers n (i.e., those n for which $n \neq n_k, m_k$ for every k). The permutation π is well-defined since $n_i \neq m_j$ for every i, j . Let Z_π denote the set $\{n: n = n_k \text{ or } n = m_k \text{ for some } k\}$. Hence $\pi(n) = n$ for all $n \notin Z_\pi$. Then

$$\begin{aligned} \sum_n a_n x_{\pi(n)} &= \sum_{n \notin Z_\pi} a_n x_n + \sum_k (a_{n_k} x_{m_k} + a_{m_k} x_{n_k}) \\ &= \sum_{n \notin Z_\pi} a_n x_n + \sum_k (a_{n_k} x_{n_k} + a_{m_k} x_{m_k}) + (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) \\ &= \sum_n a_n x_n + \sum_k (a_{n_k} - a_{m_k})(x_{m_k} - x_{n_k}) \\ &= a \cdot x + \varepsilon = r, \end{aligned}$$

and so $r \in S(a, x)$, which proves (3).

Q.E.D.

THEOREM 2. Let $a = (a_n)$ where $a_1 > 0$ and $a_n \uparrow \infty$. Then a_{n+1}/a_n is bounded if and only if, for every $x = (x_n)$ for which $x_n \downarrow 0$, $S(a, x) = [a \cdot x, \infty]$.

PROOF. If a_{n+1}/a_n is bounded, then by Theorem 1, if $x_n \downarrow 0$, then $x = (x_n)$ satisfies condition (2) of the theorem. Also by Theorem 1, since $a_n \uparrow \infty$ and $a_1 > 0$, condition (3) of the theorem is satisfied by x . That is, $S(a, x) = [a \cdot x, \infty]$.

Conversely, if $S(a, x) = [a \cdot x, \infty]$ for every $x = (x_n)$ for which $x_n \downarrow 0$, we claim that a_{n+1}/a_n must remain bounded.

Suppose to the contrary that a_{n+1}/a_n is not bounded. Let $h(n)$ denote the least positive integer k for which $k \geq n$ and $a_{k+1}/a_k \geq 4^n$. Clearly $h(n)$ is a non-decreasing function of n . Define $x_n = (a_{h(n)} 3^n)^{-1}$. Then $x_n \downarrow 0$. Letting $x = (x_n)$, we claim that $S(a, x) \neq [a \cdot x, \infty]$. In fact, we claim that $a \cdot x < 1$ but $1 \notin S(a, x)$. Indeed, $a \cdot x = \sum_n a_n x_n = \sum_n (a_{h(n)} 3^n)^{-1} \leq \sum 3^{-n} = 1/2 < 1$. Furthermore, letting π be any permutation of Z^+ , if $\pi^{-1}(k) > h(k)$ for some k , then

$$\begin{aligned} \sum_n a_n x_{\pi(n)} &\geq a_{\pi^{-1}(k)}^{-1} x_k \geq a_{h(k)+1} x_k = a_{h(k)+1} (a_{h(k)} 3^k)^{-1} \\ &\geq 4^k 3^{-k} > 1. \end{aligned}$$

On the other hand, if $\pi^{-1}(k) \leq h(k)$ for every k , then

$$\sum_n a_n x_{\pi(n)} = \sum_{\pi^{-1}(n)} a_{\pi^{-1}(n)} x_n \leq a_{h(n)} x_n = \sum 3^{-n} = 1/2 < 1.$$

In any case, $\sum_n a_n x_{\pi(n)} \neq 1$, hence $1 \notin S(a, x)$.

Q.E.D.

NOTE. In the proof of Theorem 1, each time we constructed a permutation π to solve the equation $\sum_n a_n x_{\pi(n)} = r$, it sufficed to use only disjoint 2-cycles. That is, each such π that we constructed was the product of disjoint 2-cycles. This seems odd and leads us to ask if there are any circumstances in which the use of infinite-cycles or n -cycles yields more. In other words, is it always true that $S(a, x)$ is the same as $\{\sum_n a_n x_{\pi(n)} : \pi \text{ is a permutation of } \mathbb{Z}^+ \text{ which is a product of disjoint 2-cycles}\}$?

The following question seems likely to have an affirmative answer. If so, this would give a characterization for those sequences a and x where $a_n \uparrow \infty$, $a_1 > 0$, and $x_n \uparrow 0$, which satisfy $S(a, x) = [a \cdot x, \infty]$. However, it remains unsolved.

QUESTION 1. If a and x are as above, does (3) $\implies$ (2) in Theorem 1?

Finally, we wish to point out that Theorems 1 and 2 imply analogous theorems in which a and x switch roles. Indeed, the proofs of the following two corollaries follow naturally along the same lines as those of Theorems 1 and 2.

COROLLARY 3. Let $x = (x_n)$ where $x_n > 0$ for all n , and $x_n \rightarrow 0$ as $n \rightarrow \infty$. Consider the following conditions.

(1) x_n/x_{n+1} is bounded below.

(2) For the non-negative sequence $a = (a_n)$, there exist subsequences (a_{n_k})

and (x_{m_k}) of a and x , respectively, such that

a) $a_{n_k} x_{m_k} \rightarrow 0$ as $k \rightarrow \infty$, and

b) $\sum_k a_{n_k} x_{m_k} = \infty$.

Then (1) implies that (2) holds for every strictly positive sequence $a = (a_n)$ that tends to ∞ .

COROLLARY 4. Let $x = (x_n)$ be a non-negative sequence. Then x_n/x_{n+1} is bounded below if and only if, for every $a = (a_n)$ for which $a_n \uparrow \infty$ and $a_1 > 0$, $S(a, x) = [a \cdot x, \infty]$.

QUESTION 2. Is there anything to be said about the qualitative nature of $S(a, x)$? Is it always a Borel set, measurable, F_σ , G_σ ?

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