SOME RADIUS OF CONVEXITY PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper we consider some radius of convexity problems for certain classes of analytic functions. These classes, in general, are related with functions of bounded boundary rotation.

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1. INTRODUCTION.

Let \( V_k \) be the class of functions of bounded boundary rotation. Patero [1] showed that a function \( f \), analytic in \( E = \{ z : |z| < 1 \} \), \( f(0) = 0 \), \( f'(0) = 1 \), \( f'(z) \neq 0 \); is in \( V_k \) if and only if, for \( z = re^{i\theta} \),

\[
\int_0^{2\pi} \left| \text{Re} \left( \frac{z f'(z)}{f'(z)} \right) \right| \, d\theta \leq k \pi
\]

It is geometrically obvious that \( k \geq 2 \) and \( V_2 \subseteq \mathbb{C} \), the class of univalent convex function.

A class \( T_k \) of analytic functions related with the class \( V_k \) has been introduced and discussed in [2]. Let \( f \) with \( f(0) = 0 \), \( f'(0) = 1 \) be analytic in \( E \). Then \( f \in T_k \), \( k \geq 2 \), if there exists a function \( g \in V_k \) such that, for \( z \in E \),

\[
\text{Re} \left( \frac{f'(z)}{g'(z)} \right) > 0
\]

It is clear that \( T_2 \equiv K \), the class of close-to-convex functions introduced by Kaplan [3].

Let \( P_{\alpha, n} \) denote the class of functions \( p(z) \) in \( E \) given by

\[
p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots, \quad n \geq 1,
\]

which satisfy the inequality
\[ |p(z) - \frac{1}{\alpha a}| < \frac{1}{2a}, \quad 0 < \alpha < 1. \]

The class \( P_{\alpha, n} \) has been introduced in [4]. If \( \alpha = 0 \), the class \( P_{\alpha, n} \) reduces to the classical class of functions with positive real part.

We shall need the following results in the next section.

**Lemma 1.1** [4]. Let \( p \in P_{\alpha, n} \), then for \( z \in E, \ |z| < r < 1 \)

(i) \[ \frac{1 - r^n}{1 + cr^n} \leq \frac{p(z)}{|p(z)|} \leq \frac{1 + r^n}{1 - cr^n} \]

(ii) \[ \frac{p'(z)}{|p(z)|} \leq \frac{(1+z)n r^{n-1}}{(1+cr^n)(1-r^n)} \]

where \( c = 1 - 2\alpha \).

**Lemma 2.2** [5]. If \( u \) and \( D \) are analytic in \( E \) and \( u(0) = D(0) = 0 \), and if \( D \) maps \( E \) onto many-sheeted region, which is starlike with respect to the origin, then \( \Re \frac{u'}{D'} > 0 \Rightarrow \Re \frac{u}{D} > 0, \ z \in E. \)

**Lemma 2.3** [6]. Let \( g \in V_k \). Then \( G(z) = \frac{2}{z} \int_0^z g(t) dt \) is convex in the disc \( |z| < \frac{1}{2} (1 - \sqrt{k-1}) \).

2. MAIN RESULTS

In all of the theorems, \( f \) and \( g \) will be analytic in \( E \), \( f'(0) = 1 \), \( f(0) = 0 \). The univalence will not be assumed unless explicitly stated.

**Theorem 2.1.** Let \( g \in V_k \) and let \( \frac{f'(z)}{g'(z)} \in P_{\alpha, 1} \). Then \( f \) maps \( |z| < r \) onto a convex domain, where \( r \) is the least positive root of

\[ cx^2 - x^2(c + 2) - x(k+1) + 1 = 0. \]  

**Proof:** Let \( \frac{f'(z)}{g'(z)} = p(z), \ p(z) \in P_{\alpha, 1} \).

Then

\[ \frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zp'(z)}{p(z)}. \]

Hence

\[ \Re \left( \frac{(zf'(z))'}{f'(z)} \right) \geq \Re \left( \frac{(zg'(z))'}{g'(z)} \right) - \left| \frac{zp'(z)}{p(z)} \right|. \]  

Now, it is known [7] that if \( g \in V_k \), then

\[ \Re \left( \frac{(zf'(z))'}{f'(z)} \right) \geq \frac{2 - kr + 1}{1 - r^2}. \]  

Using (2.3) and Lemma 1.1(1) for \( n = 1 \), (2.2) becomes
Thus \( f \) is convex if the right hand side of (2.1) is positive.

**Corollary 2.1.** Let \( \alpha = 0 \) \((c = 1)\) which means \( \Re \frac{f'(z)}{G'(z)} > 0 \). Then \( f \) maps 
\[ |z| < r = \frac{(k+2) - \sqrt{k^2 + 4k}}{2} \]
onto a convex domain. This result was obtained in [2].

**Corollary 2.2.** For \( \alpha = \frac{1}{2} \), we have \( \Re \frac{f'(z)}{G'(z)} - 1 < 1 \). Then \( f \) is convex for
\[ |z| < r = \frac{1}{k+1} \]. For \( k = 4 \), \( V_4 \) consists of univalent functions and \( r = \frac{1}{2} \).
This result is known [4].

**Corollary 2.3.** Let \( \alpha = 0 \), and \( k = 2 \), then \( f \) maps \( |z| < r = 2 - \sqrt{2} \) onto a convex domain. This result is well-known [8].

**Remarks 2.1.** Let \( \alpha = 0 \) and \( k = 1 \). Then we obtain the known result \( r = 3 - 2\sqrt{2} \)
of Ratti [9].

**Theorem 2.2.** Let \( G \in T_k \) and let \( \frac{f'(z)}{G'(z)} \in P_{1,1} \). Then \( f \) maps \( |z| < r \) onto a convex domain where \( r \) is the least positive root of the equation
\[ cx^3 - (k+5)cx^2 - (k+3)cx + 1 = 0. \]

**Proof:** Let \( \frac{f'(z)}{G'(z)} = p(z) \), where \( p(z) \in P_{1,1} \), \( g \in T_k \).

Then
\[ \Re \left( \frac{zf'(z)}{f'(z)} \right) > \Re \left( \frac{z^{-1}f'(z)}{G'(z)} \right) - \left| \frac{zp'(z)}{p(z)} \right| \]

For \( g \in T_k \), it is known [2] that
\[ \Re \left( \frac{zf'(z)}{G'(z)} \right) > \frac{r^2 - (k+2)r + 1}{1 - r^2}. \]

Using (2.4) and Lemma 1.1(ii), we obtain the result.

**Corollary 2.4.** Let \( \alpha = \frac{1}{2} \) \((c = 0)\) and in this case \( f \) maps \( |z| < r = \frac{1}{k+1} \) onto a convex domain. The special case for \( k = 2 \) is known [4].

**Corollary 2.5.** For \( \alpha = 0 \) and \( k = 2 \), \( T_{2} \subseteq K \) consists of close-to-convex univalent functions. Then the radius of convexity is \( r = 3 - 2\sqrt{2} \). This result is known [9].

**Theorem 2.3.** Let \( \Re \frac{f'(z)}{G'(z)} > 0 \) and \( \Re \frac{p'(z)}{S'(z)} > 0 \), where \( S \) belongs to the
class $S^*$ of starlike functions. Then $f$ maps $|z| < r = 4 - \sqrt{15}$ onto a convex domain.

\textbf{Proof:} We have

$$f'(z) = S'(z)h_1(z)h_2(z), \quad \text{where} \quad \text{Re } h_1(z) > 0, \ \text{Re } h_2(z) > 0, \ \forall z \in \mathbb{E}.$$ 

That is

$$\frac{(z f'(z))'}{f'(z)} = \frac{(z S'(z))'}{S'(z)} + \frac{zh_1'(z)}{h_1(z)} + \frac{zh_2'(z)}{h_2(z)}$$

hence

$$\text{Re} \left( \frac{(z f'(z))'}{f'(z)} \right) > \text{Re} \left( \frac{(z S'(z))'}{S'(z)} \right) - \frac{|zh_1'(z)|}{h_1(z)} - \frac{|zh_2'(z)|}{h_2(z)}. \quad (2.5)$$

Now it is well known [8] that for $S \in S^*$,

$$\text{Re} \left( \frac{(z S'(z))'}{S'(z)} \right) > \frac{1-4r + r^2}{1-r^2}. \quad (2.6)$$

Also, if $\text{Re } h(z) > 0$, then it is known [11] that

$$\left| \frac{zh_1'(z)}{h(z)} \right| < \frac{3r}{1-r^2}. \quad (2.7)$$

Using (2.6) and (2.7), (2.5) yield:

$$\text{Re} \left( \frac{(z f'(z))'}{f'(z)} \right) > \frac{1-8r + r^2}{1-r^2}.$$

Hence $f$ is convex for $|z| < r = 4 - \sqrt{15}$.

**Theorem 2.4.** Let $\text{Re } \frac{f'(z)}{g'(z)} > 0$ and $\text{Re } \frac{S'(z)}{g'(z)} > 0$ where $S \in \mathbb{T}_k$.

Then $f$ maps $|z| < r = (k+6) - \sqrt{(k+6)^2 - 4}$ onto a convex domain.

The proof follows on the same lines of Theorem 2.3, by using (2.4).

**Corollary 2.6.** If $k = 2$, then $S \in \mathbb{T}_2 = \mathbb{K}$. In this case $f$ maps $|z| < r = 4 - \sqrt{15}$ onto a convex domain.

**Theorem 2.5.** Let $f \in \mathbb{V}$ and $f_a(z) = \int_0^z (f'(t))^a \text{d}t, \ \forall a \in \mathbb{I}$. Then $f_a$ maps $|z| < r$ onto a convex domain, where $r$ is the least positive root of

$$(2a-1)x^2 - ax + 1 = 0. \quad (2.8)$$

\textbf{Proof:} We have $f'(z) = (f'(z))^a, \ \forall a \in \mathbb{I}$.

Thus

$$\frac{(zf'(z))'}{f'(z)} = a \frac{(zf'(z))'}{f'(z)} + (1-a).$$
Since \( f e V_k \), using (2.3), we have

\[
\Re \frac{(zf'(z))^\alpha}{f'(z)} = \alpha \frac{1 - kr + r^2}{1 - r^2} + (1 - \alpha) = \frac{1 - akr + (\alpha - 1)r^2}{1 - r^2}
\]

and this gives us the required result.

**Theorem 2.6.** Let \( f e T_k \) and \( f'(z) = \int_0^z (f'(t))^2 dt \). Then \( f' \) maps \( |z|<r \) onto a close-to-convex domain, where \( r \) is the least positive root of (2.8).

**Proof:** Since \( f e T_k \), there exists \( \omega e V_k \) such that \( \Re \frac{f'(z)}{g'(z)} > 0 \). Let \( g'(z) = \int_0^z |g'(t)|^2 dt \). Then

\[
(f'(z)/g'(z)) = |f'(z)/g'(z)|^\alpha.
\]

Using theorem 2.5, it follows that \( f' \) is close-to-convex for \( |z|<r \), where \( r \) is the least positive root of (2.8).

**Corollary 2.7.** Let \( f e T_k \), then \( f' \) is close-to-convex for \( |z|<r \), where \( r \) is the least positive root of

\[
(2a-1)x^2 - 4a + 1 = 0.
\]

In this case, if \( \alpha = \frac{1}{2} \), then \( f' \) is close-to-close for \( |z|<\frac{1}{2} \).

**Corollary 2.8.** Let \( f e T_k \) and \( \alpha = \frac{1}{2} \). Then \( f' \) is close-to-convex for \( |z|<r = \frac{2}{k} \). For \( k=2 \), we have a result proved in [11].

**Corollary 2.9.** For \( k=2 \), \( f' e T_k \), see [11].

**Theorem 2.7.** Let \( f e T_k \) and \( F(z) = \int_0^z f(t) dt \). Then \( F \) maps \( |z|<r = \frac{1}{2} (k - \sqrt{k^2 - 4}) \) onto a close-to-convex domain.

**Proof:** Since \( f e T_k \), there exists \( g e V_k \) such that \( \Re \frac{f'(z)}{g'(z)} > 0 \). Let \( G(z) = \int_0^z g(t) dt \). We know, from Lemma 1.3, that \( G \) is convex for \( |z|<r = \frac{k - \sqrt{k^2 - 4}}{2} \). Now

\[
\frac{F'(z)}{G'(z)} = \frac{\int_0^z f(t) dt'}{\int_0^z g(t) dt'} = \frac{N}{D},
\]

and

\[
\frac{N'}{D'} = \frac{f'(z)}{g'(z)}.
\]
So \( \text{Re} \frac{N}{D'} > 0 \). Applying Lemma 1.2 for \( |z| < r = \frac{k- \sqrt{k^2 - 4}}{2} \), we have \( \text{Re} \frac{N}{D'} > 0 \), which implies that \( F \) is close-to-convex for
\[ |z| < r = \frac{k- \sqrt{k^2 - 4}}{2} . \]

**Corollary 2.10.** When \( k = 2 \), \( f \in \mathbb{T}_2 \subseteq K \) and hence \( f \in K \) for \( z \in E \). This result was obtained in [5] by Libera.

**REFERENCES**


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