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SOME RADIUS OF CONVEXITY PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

KHALIDA I. NOOR AND HAILAH AL-MADIFER

Mathematics Department
Science College of Education for Girls
Sitten Road, Riyadh, Saudi Arabia

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ABSTRACT. In this paper we consider some radius of convexity problems for certain classes of analytic functions. These classes, in general, are related with functions of bounded boundary rotation.

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1. INTRODUCTION.

Let V_k be the class of functions of bounded boundary rotation. Paatero [1] showed that a function f, analytic in $E = \{z:|z|<1\}, f(0) = 0, f'(0)=1, f'(z) \neq 0$; is in V_k if and only if, for $z = re^{i\theta}$,

$$\int_0^{2\pi} \left|\mathrm{Re}\, \frac{(zf'(z))'}{f'(z)}\right| d\theta \leq k\pi$$

It is geometrically obvious that $k\geq 2$ and $V_2 \equiv C$, the class of univalent convex function.

A class T_k of analytic functions related with the class V_k has been introduced and discussed in [2]. Let f with $f(0) = 0$, $f'(0) = 1$ be analytic in E. Then $f\varepsilon T_k$, $k\geq 2$, if there exists a function $g\varepsilon V_k$ such that, for $z\varepsilon E$,

$$\mathrm{Re}\, \frac{f'(z)}{g'(z)} > 0$$

It is clear that $T_2 \equiv K$, the class of close-to-convex functions introduced by Kaplan [3].

Let $P_{\alpha,n}$ denote the class of functions $p(z)$ in E given by $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots\ldots$, $n \geq 1$, which satisfy the inequality

$$|p(z) - \frac{1}{2\alpha}| < \frac{1}{2\alpha}, \quad 0 \leq \alpha < 1 .$$

The class $P_{\alpha,n}$ has been introduced in [4]. If $\alpha=0$, the class $P_{\alpha,n}$ reduces to the classical class of functions with positive real part.

We shall need the following results in the next section.

Lemma 1.1[4]. Let $p \in P_{\alpha,n}$, then for $z \in E$, $|z| = r < 1$

(i) $$\frac{1 - r^n}{1 + cr^n} \leq \operatorname{Re} p(z) \leq |p(z)| \leq \frac{1 + r^n}{1 - cr^n}$$

(ii) $$\left|\frac{p'(z)}{p(z)}\right| \leq \frac{(1+c)n\, r^{n-1}}{(1+cr^n)(1-r^n)},$$

where $c = 1-2\alpha$.

Lemma 1.2 [5]. If N and D are analytic in E and $N(0) = D(0) = 0$, and if D maps E onto many-sheeted region, which is starlike with respect to the origin, then $\operatorname{Re} \frac{N'}{D'} > 0 \Rightarrow \operatorname{Re} \frac{N}{D} > 0$, $z \in E$.

Lemma 1.3 [6]. Let $g \in V_k$. Then $G(z) = \frac{2}{z}\int_0^z g(t)dt$ is convex in the disc $|z| < \frac{1}{2}(k - \sqrt{k^2-4})$.

2. MAIN RESULTS

In all of the theorems, f and g will be analytic in E, $f'(0) = 1$, $f(0) = 0$. The univalence will not be assumed unless explicitly stated.

THEOREM 2.1. Let $g \in V_k$ and let $\frac{f'(z)}{g'(z)} \in P_{\alpha,1}$. Then f maps $|z| < r$ onto a convex domain, where r is the least positive root of

$$cx^3 - x^2(ck+c) - x(k+1) + 1 = 0 . \tag{2.1}$$

PROOF: Let $\frac{f'(z)}{g'(z)} = p(z)$, $p(z) \in P_{\alpha,1}$.

Then

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zp'(z)}{p(z)} .$$

Hence

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} \geq \operatorname{Re} \frac{(zg'(z))'}{g'(z)} - \left|\frac{zp'(z)}{p(z)}\right| . \tag{2.2}$$

Now, it is known [7] that if $g \in V_k$, then

$$\operatorname{Re} \frac{(zg'(z))'}{g'(z)} \geq \frac{r^2 - kr + 1}{1 - r^2} . \tag{2.3}$$

Using (2.3) and Lemma 1.1(ii) for $n=1$, (2.2) becomes

$$\operatorname{Re}\frac{(zf'(z))'}{f'(z)} \geq \frac{cr^3-(k+1)cr^2-(k+1)r+1}{(1-r^2)(1+cr)}$$

Thus f is convex if the right hand side of (2.1) is positive.

Corollary 2.1. Let $\alpha=0$ $(c=1)$ which means $\operatorname{Re}\frac{f'(z)}{g'(z)}>0$. Then f maps $|z|<r=\frac{(k+2)-\sqrt{k^2+4k}}{2}$ onto a convex domain. This result was obtained in [2].

Corollary 2.2. For $\alpha=\frac{1}{2}$, we have $\left|\frac{f'(z)}{g'(z)}-1\right|<1$. Then f is convex for $|z|<r=\frac{1}{k+1}$. For $k=4$, V_4 consists of univalent functions and $r=\frac{1}{5}$. This result is known [4].

Corollary 2.3. If $\alpha=0$, and $k=2$, then f maps $|z|<r=2-\sqrt{3}$ onto a convex domain. This result is well-known [8].

REMARKS 2.1. Let $\alpha=0$ and $k=4$. Then we obtain the known result $r=3-2\sqrt{2}$ of Ratti [9].

THEOREM 2.2. Let $g\in T_k$ and let $\frac{f'(z)}{g'(z)}\in P_{\alpha,1}$. Then f maps $|z|<r$ onto a convex domain where r is the least positive root of the equation

$$cx^3-(k+3c)x^2-(k+3)x+1=0 .$$

PROOF: Let $\frac{f'(z)}{g'(z)}=p(z)$, where $p(z)\in P_{\alpha,1}$, $g\in T_k$.

Then

$$\operatorname{Re}\frac{(zf'(z))'}{f'(z)} \geq \operatorname{Re}\frac{(zg'(z))'}{g'(z)} - \left|\frac{zp'(z)}{p(z)}\right|$$

For $g\in T_k$, it is known [2] that

$$\operatorname{Re}\frac{(zg'(z))'}{g'(z)} \geq \frac{r^2-(k+2)r+1}{1-r^2} . \tag{2.4}$$

Using (2.4) and Lemma 1.1(ii), we obtain the result.

Corollary 2.4. Let $\alpha=\frac{1}{2}$ $(c=0)$ and in this case f maps $|z|<r=\frac{1}{k+3}$ onto a convex domain. The special case for $k=2$ is known [4].

Corollary 2.5. For $\alpha=0$ and $k=2$, $T_2\equiv K$ consists of close-to-convex univalent functions. Then the radius of convexity is $r=3-2\sqrt{2}$. This result is known [9].

THEOREM 2.3. Let $\operatorname{Re}\frac{f'(z)}{g'(z)}>0$ and $\operatorname{Re}\frac{g'(z)}{S'(z)}>0$, where S belongs to the

class S^* of starlike functions. Then f maps $|z| < r = 4 - \sqrt{15}$ onto a convex domain.

PROOF: We have

$$f'(z) = S'(z)h_1(z)h_2(z), \quad \text{where } \operatorname{Re} h_1(z)>0, \ \operatorname{Re} h_2(z)>0, \quad z\in E.$$

That is

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zS'(z))'}{S'(z)} + \frac{zh_1'(z)}{h_1(z)} + \frac{zh_2'(z)}{h_2(z)}$$

hence

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} \geq \operatorname{Re} \frac{(zS'(z))'}{S'(z)} - \left|\frac{zh_1'(z)}{h_1(z)}\right| - \left|\frac{zh_2'(z)}{h_2(z)}\right| . \tag{2.5}$$

Now it is well known [8] that for $S\in S^*$,

$$\operatorname{Re} \frac{(zS'(z))'}{S'(z)} \geq \frac{1-4r+r^2}{1-r^2} . \tag{2.6}$$

Also, if $\operatorname{Re} h(z) > 0$, then it is known [10] that

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2r}{1-r^2} . \tag{2.7}$$

Using (2.6) and (2.7), (2.5) yields

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} \geq \frac{1-8r+r^2}{1-r^2} .$$

Hence f is convex for $|z| < r = 4 - \sqrt{15}$.

THEOREM 2.4. Let $\operatorname{Re} \frac{f'(z)}{g'(z)} > 0$ and $\operatorname{Re} \frac{g'(z)}{S'(z)} > 0$ where $S\in T_k$.

Then f maps $|z| < r = \frac{(k+6) - \sqrt{(k+6)^2-4}}{2}$ onto a convex domain.

The proof follows on the same lines of Theorem 2.3, by using (2.4).

Corollary 2.6. If $k=2$, then $S\in T_2 \equiv K$. In this case f maps $|z|<r=4-\sqrt{15}$ onto a convex domain.

THEOREM 2.5. Let $f\in V_k$ and $f_\alpha(z) = \int_0^z (f'(t))^\alpha dt$, $0\leq\alpha\leq 1$. Then f_α maps $|z|<r$ onto a convex domain, where r is the least positive root of

$$(2\alpha-1)\, x^2 - \alpha k x + 1 = 0 . \tag{2.8}$$

PROOF: We have $f'_\alpha(z) = (f'(z))^\alpha$, $0\leq\alpha\leq 1$.

Thus

$$\frac{(zf'_\alpha(z))'}{f'_\alpha(z)} = \alpha \frac{(zf'(z))'}{f'(z)} + (1-\alpha) .$$

Since $f\varepsilon V_k$, using (2.3), we have

$$\operatorname{Re}\frac{(zf'_\alpha(z))'}{f'_\alpha(z)} \geq \alpha\,\frac{1-kr+r^2}{1-r^2} + (1-\alpha) = \frac{1-\alpha kr+(2\alpha-1)r^2}{1-r^2}$$

and this gives us the required result.

<u>THEOREM 2.6</u>. Let $f\varepsilon T_k$ and $f_\alpha(z) = \int_0^z (f'(t))^\alpha dt$. Then f_α maps $|z|<r$ onto a close-to-convex domain, where r is the least positive root of (2.8).

<u>PROOF</u>: Since $f\varepsilon T_k$, there exists $g\varepsilon V_k$ such that $\operatorname{Re}\frac{f'(z)}{g'(z)} > 0$. Let $g_\alpha(z) = \int_0^z [g'(t)]^\alpha dt$. Then

$$(f'_\alpha(z)/g'_\alpha(z)) = (f'(z)/g'(z))^\alpha .$$

Using theorem 2.5, it follows that f_α is close-to-convex for $|z|<r$, where r is the least positive root of (2.8).

<u>Corollary 2.7</u>. Let $f\varepsilon T_4$, then f_α is close-to-convex for $|z|<r$, where r is the least positive root of

$$(2\alpha-1)x^2 - 4\alpha x + 1 = 0 .$$

In this case, if $\alpha = \frac{1}{2}$, then f_α is close-to-close for $|z| < \frac{1}{2}$.

<u>Corollary 2.8</u>. Let $f\varepsilon T_k$ and $\alpha = \frac{1}{2}$. Then f_α is close-to-convex for $|z|<r = \frac{2}{k}$. For $k=2$, we have a result proved in [11].

<u>Corollary 2.9</u>. For $k=2$, $f_\alpha \varepsilon K$, see [11].

<u>THEOREM 2.7</u>. Let $f\varepsilon T_k$ and $F(z) = \frac{2}{z}\int_0^z f(t)dt$. Then F maps $|z|<r = \frac{1}{2}(k - \sqrt{k^2-4})$ onto a close-to-convex domain.

<u>PROOF</u>: Since $f\varepsilon T_k$, there exists a $g\varepsilon V_k$ such that $\operatorname{Re}\frac{f'(z)}{g'(z)} > 0$. Let $G(z) = \frac{2}{z}\int_0^z g(t)dt$. We know, from Lemma 1.3, that G is convex for $|z| < r = \frac{k-\sqrt{k^2-4}}{2}$. Now

$$\frac{F'(z)}{G'(z)} = \frac{\left(\frac{2}{z}\int_0^z f(t)dt\right)'}{\left(\frac{2}{z}\int_0^z g(t)dt\right)'} = \frac{N}{D}$$

and

$$\frac{N'}{D'} = \frac{f'(z)}{g'(z)} .$$

So $\operatorname{Re} \frac{N'}{D'} > 0$. Applying Lemma 1.2 for $|z| < r = \frac{k-\sqrt{k^2-4}}{2}$

we have $\operatorname{Re} \frac{N}{D} > 0$, which implies that F is close-to-convex for

$|z| < r = \frac{k-\sqrt{k^2-4}}{2}$.

Corollary 2.10. When $k=2$, $f \in T_2 \equiv K$ and hence $F \in K$ for $z \in E$. This result was obtained in [5] by Libera.

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