ON THE EXISTENCE OF EQUATIONS OF EVOLUTION

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ABSTRACT. A time-independent, non-autonomous non-linear system governed by a principle of determinism (the state at a given time is determined by the initial state and by the control history during the intervening closed interval) is shown to obey a generalized evolution equation (1.2), where n is such that the state is continuously differentiable with respect to time whenever the control is of class $C^n$.

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1. INTRODUCTION.

The equations of evolution of time-independent, non-autonomous non-linear systems are almost universally taken as

$$u' = f(u, x)$$  \hspace{1cm} (1.1)

where $u$ is the output (or state) and $x$ is the input (or control), both functions of time $t$ with values in finite-dimensional vector spaces (say $U$ and $X$, respectively), and the prime denotes differentiation with respect to time.

However, equation (1.1) is not the most general evolution equation. It may be regarded as a special case (corresponding to $n = 0$) of

$$u' = f(u, x, x', \ldots, x^{(n)}).$$  \hspace{1cm} (1.2)

An example of the need for an equation of the form (2) occurs in the mechanics of inelastic continua. Here $u$ is the vector whose components are the internal variables (or the inelastic strain), while $x$ is stress or strain. Viscoelastic and viscoplastic materials are described by equation (1.1). Plastic materials, on the other hand, require equation (1.2) with $n = 1$. In particular, for a rate-independent material the function $f$ must be first-degree homogeneous in $x'$.

The difference between systems described by equations (1.1) and (1.2) (or, more generally, by equations (1.2) with different values of $n$) lies in the character of the solutions $u(t)$. If $f$ is continuous, then the solution of (1.1) is continuously differentiable whenever $x$ is a continuous function of time. On the other hand, solution
of (1.2) will not in general be continuously differentiable unless \( x \) is of class \( C^N \).

In other words, the choice of \( n \) in equation (1.2) depends on the way in which the system smooths the input: the greater the smoothing, the lower the value of \( n \).

However, the existence of an equation of evolution cannot be assumed a priori for an arbitrary system. In this note we shall try to find sufficient conditions for the existence of an equation of evolution, and to relate the value of \( n \) to the smoothing property of the system.

2. MAIN RESULTS

We shall assume the system to be governed by a principle of determinism as follows: the value of \( u \) at time \( t+\tau \) is determined by its value at time \( t \) and by the history of \( x \) during the interval \([t, t+\tau]\). We shall express this mathematically as follows. Let \( x^t \) be defined by \( x^t(s) = x(t+s) \), and let \( x^t_T \) denote the restriction of \( x^t \) to \([0, \tau]\). The pair \((u(t), x^t_T)\) may be regarded as determining a process of duration \( \tau \) in the system; let \( P^n_T \) denote the set of all such pairs determining possible processes in the system. Then there exists a mapping \( \phi^T : P^n_T \rightarrow U \) such that

\[
 u(t+\tau) = \phi^T(t, u(t), x^t_T) \tag{2.1}
\]

Furthermore, let \( C^n_T \) denote the Banach space \( C^n([0, \tau]; X) \), let \( P^n_T = P^n_T(U \times C^n_T) \). Then the smoothing property of the system may be expressed by saying that the left-hand side of (2.1) is differentiable with respect to \( \tau \) whenever \((u(t), x^t_T) \in P^n_T \), that is, that the limit

\[
\lim_{\tau \to 0^+} \frac{1}{\tau} [\phi^T(u(t), x^t_T) - u(t)] \tag{2.2}
\]

exists and is continuous in \( t \) under that condition. It is clear that this limit is determined by \( u(t) \) and by the behavior of \( x \) in a neighborhood of \( t \). It is not immediately clear that it should take the form of the right-hand side of (1.2). In fact, in order to derive this result we need to assume some properties of the restriction to \( P^n_T \) of the mapping \( \phi^T \) as given in the following theorem.

THEOREM. Let \( \phi^T : P^n_T \rightarrow U \) be such that \( \phi^T(a, \cdot) \) is locally Lipschitz (with respect to the \( C^n_T \) norm), the local Lipschitz norm \( P^n_T \) being \( O(\tau) \) as \( \tau \to 0^+ \). Then the limit

\[
\lim_{\tau \to 0^+} \frac{1}{\tau} [\phi^T(a, y) - a],
\]

when it exists, depends only on \( a, y, y', \ldots, y^{(n)} \).

PROOF. Define \( \overline{y} \in C^n_T \) by

\[
\overline{y}(s) = y(0) + y'(0)s + \ldots + y^{(n)}(0)s^n/n!
\]

Let \( ||\cdot||_T \) denote the \( C^n_T \) norm and \( ||\cdot|| \) any finite-dimensional norm. Then

\[
||y - \overline{y}||_T = \max_{n} \sup_{[0, \tau]} |y(s) - y(0) - \ldots - y^{(n)}(0)s^n/n!|,
\]

\[
\sup_{[0, \tau]} |y'(s) - y'(0) - \ldots - y^{(n)}(0)s^{n-1}/(n-1)!|, \ldots, \sup_{[0, \tau]} |y^{(n)}(s) - y^{(n)}(0)|.
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\]
Since the argument of each $|\cdot|$ is a continuous function of $s$ that vanishes at $0$, each supremum goes to zero as $\tau \to 0$, so that $\|y - \overline{y}\|_\tau \to 0$ as $\tau \to 0$. Now

$$\frac{1}{\tau} [\phi_\tau(a,y) - a] = \frac{1}{\tau} [\phi_\tau(a,y) - \phi_\tau(a,\overline{y})] + \frac{1}{\tau} [\phi_\tau(a,\overline{y}) - a];$$

but for the first term on the right-hand side we have

$$\frac{1}{\tau} \left| \phi_\tau(a,y) - \phi_\tau(a,\overline{y}) \right| \leq \frac{F_\tau}{\tau} \|y - \overline{y}\|_\tau,$$

so that this term goes to zero as $\tau \to 0$. Consequently,

$$\lim_{\tau \to 0^+} \frac{1}{\tau} [\phi_\tau(a,y) - a] = \lim_{\tau \to 0^+} \frac{1}{\tau} [\phi_\tau(a,\overline{y}) - a],$$

and this last limit, whenever it exists, depends only on $a$ and on the parameters defining $\overline{y}$, that is, $y(0), y'(0), \ldots, y^{(n)}(0)$. Q.E.D.

By the assumed smoothing property of the system the limit exists and is a continuous function of its arguments, $f(a,y(0), y'(0), \ldots, y^{(n)}(0))$; it is this $f$ that furnishes the right-hand side of equation (2).

As a very simple example, consider $\phi_\tau$ given by

$$\phi_\tau(a,y) = a + y(\tau) - y(0).$$

This $\phi_\tau(a,\cdot)$ is Lipschitz (since it is a continuous linear mapping) on $C^0_\tau$, but the Lipschitz norm over this space is 2. On the other hand, we may rewrite it as

$$\phi_\tau(a,y) = a + \int_0^\tau y'(s) ds,$$

so that it is clearly a continuous linear mapping of $C^1_\tau$, and its (Lipschitz) norm is $\tau$. Consequently, $n = 1$, and indeed we have $f(a,y,y') = y'$.

A more sophisticated example, relevant to plasticity theory, is

$$\phi_\tau(a,y) = a + \int_0^\tau |y'(s)| ds.$$ 

This $\phi_\tau(a,\cdot)$ is not Lipschitz on $C^0_\tau$ but it is Lipschitz on $C^1_\tau$:

$$|\phi_\tau(a,y) - \phi_\tau(a,\overline{y})| = \left| \int_0^\tau (|y'(s)| - |\overline{y}'(s)|) ds \right| \leq \int_0^\tau |y'(s)| - |\overline{y}'(s)| ds$$

$$\leq \tau \sup_{s \in [0,\tau]} |y'(s)| - |\overline{y}'(s)| \leq \tau \sup_{s \in [0,\tau]} |y'(s) - \overline{y}'(s)| \leq \tau \|y - \overline{y}\|_\tau,$$

so that once again $F_\tau = \tau$.