AN APPLICATION OF HYPERGEOMETRIC FUNCTIONS TO A PROBLEM IN FUNCTION THEORY

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ABSTRACT. In some recent work in univalent function theory, Aharonov, Friedland, and Brannan studied the series, \((1 + xt)^\alpha (1 - t)^\beta = \sum_{n=0}^{\infty} A_n^{(\alpha, \beta)}(x)t^n\). Brannan posed the problem of determining \(S = \{(\alpha, \beta) : |A_n^{(\alpha, \beta)}(e^{i\theta})| < |A_n^{(\alpha, \beta)}(1)|, 0 < \theta < 2\pi, \alpha > 0, \beta > 0, n = 1, 2, 3, \ldots\}\). Brannan showed that if \(\beta \geq \alpha > 0\), and \(\alpha + \beta \geq 2\), then \((\alpha, \beta) \in S\). He also proved that \((\alpha, 1) \in S \) for \(\alpha \geq 1\). Brannan showed that for \(0 < \alpha < 1 \) and \(\beta = 1\), there exists a \(\theta\) such that \(|A_{2k}^{(\alpha, 1)}(e^{i\theta})| > |A_{2k}^{(\alpha, 1)}(1)|\) for \(k\) any integer. In this paper, we show that \((\alpha, \beta) \in S \) for \(\alpha > 1\) and \(\beta > 1\).

KEY WORDS AND PHRASES. Hypergeometric Functions, Jacobi Polynomials, Maximum property, and positive maximum property.

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1. INTRODUCTION.

Let \(D\) be a disk \(\{z : |z - a| \leq r\}\) where the center \(a\) is real. Let \(f\) be a function analytic in an open neighborhood of the disk \(D\). It is well known that the maximum modulus of \(F\) on \(D\) is attained on the boundary \(\{z : |z - a| = r\}\). If the maximum modulus is attained at \(a + r\) and only at \(a + r\) then we say that \(f\) has the maximum property on \(D\). If in addition \(f(a + r) > 0\), then \(f\) has the positive maximum property. If the disk \(D\) is not specified then it is assumed that \(D\) is the unit disk.

Let \((1 + zt)^\alpha (1 - t)^\beta = \sum_{k=0}^{\infty} A_k^{(\alpha, \beta)}(z)t^k\) and let \(MP = \{(\alpha, \beta) : \alpha > 0, \beta > 0\} \) and \(A_n^{(\alpha, \beta)}(z)\) satisfies the positive maximum property for \(n = 1, 2, 3, \ldots\). The main problem in this paper is to characterize the sets \(MP\) and \(PMP\). An application to extreme point theory is given in [2].
2. SOME FUNDAMENTAL RECURRENCE RELATIONS

Starting with

\[(1 + at)^\alpha (1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha, \beta)}(z)t^n, \tag{2.1}\]

one can derive a number of recurrence relations. For example

\[(1 + zt)^{\alpha+\gamma} (1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma, \beta)}(z)t^n.\]

Indeed, since \((1 + zt)^{\gamma} = \sum_{n=0}^{\infty} \frac{(-\gamma)_n}{n!}(-zt)^n\), by taking the Cauchy product of this last series and the series in (2.1) we obtain

\[\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(-\gamma)_k}{k!} \frac{(-z)_k}{k!} \right) A_{n-k}^{(\alpha, \beta)}(z)t^n = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma, \beta)}(z)t^n.\]

Hence

\[A_n^{(\alpha+\gamma, \beta)}(z) = \sum_{k=0}^{n} \frac{(-\gamma)_k}{k!} \frac{(-z)_k}{k!} A_{n-k}^{(\alpha, \beta)}(z). \tag{2.2}\]

Similarly

\[A_n^{(\alpha, \beta+\gamma)}(z) = \sum_{k=0}^{n} \frac{(\gamma)_k}{k!} A_{n-k}^{(\alpha, \beta)}(z). \tag{2.3}\]

If we let \(\gamma = 1\) in (2.2), we obtain

\[A_n^{(\alpha, \beta)}(z) + zA_{n-1}^{(\alpha, \beta)}(z) = A_{n+1}^{(\alpha+1, \beta)}(z). \tag{2.4}\]

Relations (2.3) and (2.4) are significant because if \((\alpha, \beta) \in \text{PMP}\), then \((\alpha, \beta') \in \text{PMP}\) for all \(\beta' > \beta\). Also, \((\alpha, \beta) \in \text{PMP}\) implies that \((\alpha + n, \beta) \in \text{PMP}\), \(n = 1, 2, 3, \ldots\).

3. SOME EXPLICIT FORMULAS FOR \(A_n^{(\alpha, \beta)}(z)\)

Taking the Cauchy product of the series

\[(1 + zt)^{\alpha} = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} (-z)^n t^n, \quad \text{and}\]

\[(1 - t)^{-\beta} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} t^n, \quad \text{we have}\]

\[A_n^{(\alpha, \beta)}(z) = \sum_{k=0}^{n} \frac{(-\alpha)_k}{k!} \frac{(\beta)_k}{(n - k)!} (-z)_k. \tag{3.1}\]

Using the fact that \((n - k)! = (1)_{n-k}\) and \((a)_{n-k} = \frac{(a)_{n-k}}{(1-a-n)_k}\), we obtain

\[A_n^{(\alpha, \beta)}(z) = \frac{(\beta)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k}{(1 - \beta - n)_k} (-\alpha)_k.\]
Using \( _2F_1 \left( \frac{a}{c}, \frac{b}{c}; z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \), we obtain

\[
\left( \alpha, \beta \right)_n (z) = \frac{(\beta)_n}{n!} \_2F_1 \left( -n, -\alpha; -z \right).
\]

(3.2)

The Jacobi polynomials are defined as

\[
p_n^{(\alpha, \beta)} (z) = \frac{(\alpha + 1)_n}{n!} \_2F_1 \left( -n, n+\alpha+\beta+1; \frac{1-z}{2} \right).
\]

Hence

\[
A_n^{(\alpha, \beta)} (z) = (-)^n p_n^{(\beta-n, \beta-\alpha-1)} (2z+1).
\]

(3.3)

Replacing \( k \) by \( n-k \) in (3.1), and following the same procedure we get

\[
A_n^{(\alpha, \beta)} (z) = z^n p_n^{(\alpha-n, \beta-\alpha-1)} (1 + \frac{2}{z}).
\]

(3.4)

Using Pfaff's transformation [1, p. 64]

\[
_2F_1 \left( a, b; c; z \right) = (1-z)^{-a} \_2F_1 \left( a, c-b; \frac{z}{1-z} \right), \quad c \neq 0, -1, -2, \ldots
\]

we can write \( A_n^{(\alpha, \beta)} (z) \) as

\[
A_n^{(\alpha, \beta)} (z) = \frac{(\beta)_n}{n!} (1+z)^\alpha \_2F_1 \left( -\alpha, 1-\beta; \frac{z}{1+z} \right).
\]

(3.5)

Setting \( \beta = 1 \), we get

\[
A_n^{(\alpha, 1)} (z) = (1+z)^\alpha \left( 1 + \frac{(-)^n (n-a)}{(n+1)!} \frac{z^{n+1}}{z+1} \right) \_2F_1 \left( n+1, n+1-\alpha; \frac{z}{z+1} \right).
\]

(3.6)

4. **Some Maximaliy Properties for \( A_n^{(\alpha, \beta)} (z) \)**

It has been proven in [3] that \((\alpha, \beta) \in \text{MP}\) for \( \beta = 1 \) and \( \alpha \geq 1 \). We can now strengthen that result.

**Theorem 1.** \((\alpha, \beta) \in \text{PMP}\) for \( \alpha \geq 1 \) and \( \beta \geq 1 \).

**Proof:** It is evident from (3.2) that all coefficients of \( A_n^{(\alpha, \beta)} (z) \) are positive for \( \alpha \geq n \). So clearly \( A_n^{(\alpha, 1)} (z) \) will satisfy the positive maximum property for \( \alpha \geq n \).

The theorem follows from (2.3) upon showing that \( A_n^{(\alpha, 1)} (1) > 0 \) for \( 1 < \alpha < n \).

Assume that \( 1 < \alpha < n \). Then it follows from (3.6) that if
\[
\left| \frac{(-\alpha)^{n+1}}{(n+1)!} \frac{(\frac{1}{2})^{n+1}}{2\, F_1 \left( \frac{n+1-n+1-\alpha}{n+2}, \frac{1}{2} \right)} \right| < 1,
\]
(3.7)
then \(A_n^{(\alpha,l)}(1) > 0\).

Note that all terms of the \(2\, F_1\) in (3.7) are positive. Moreover
\[
2\, F_1 \left( ^{n+1-n+1-\alpha}_{n+2}, \frac{1}{2} \right) < 1\, F_0 \left( ^{n+1-\alpha}_{-}, \frac{1}{2} \right) = 2^{n+1-\alpha},
\]
by the binomial theorem.

Hence the left side of (3.7) is less than \(\left| (-\alpha)^{n+1} 2^{-\alpha}/(n+1)! \right|\). Let \(m\) be an integer such that \(m - 1 < \alpha < m\). Then
\[
\left| \frac{(-\alpha)^{n+1}}{(n+1)!} \right| = \frac{\left| (-\alpha)(1-\alpha) \cdots (m-\alpha-1)(m-\alpha) \cdots (n-\alpha) \right|}{(n+1)!} \leq \frac{\alpha(\alpha-1) \cdots (\alpha-m+1)(m-\alpha) \cdots (n-\alpha)}{(n+1)!} < \frac{m(m-1) \cdots 2\cdot 1 \cdot 1 \cdot 2 \cdot \cdots (n-m+1)}{(n+1)!} = \binom{n+1}{m}^{-1} < 1.
\]

Consequently \(\left| (-\alpha)^{n+1} 2^{-\alpha}/(n+1)! \right| < 1\), and (3.7) is established. Brannan [3], showed that \((\alpha,1) \in MP\) for \(\alpha > 1\). Hence \((\alpha,1) \in PMP\) for all \(\alpha > 1\), and by (2.3), \((\alpha,\beta) \in PMP\) for all \(\alpha > 1\) and \(\beta > 1\).

The author feels that the properties of Jacobi polynomials as given in (3.3) and (3.4) will be useful in answering other questions of Brannan's regarding the series (2.1).

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REFERENCES
