THE COMPACTUM OF A SEMI-SIMPLE COMMUTATIVE BANACH ALGEBRA

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ABSTRACT. Let A be a commutative semi-simple Banach algebra such that the set consisting of finite sums of elements from minimal left ideals coincides with that of finite sums of elements from minimal right ideals. Let $S(A)$ (the socle of A) denote this set. Let $C(A)$ denote the set of elements x in A such that the map $a \rightarrow xax$ is compact. It is shown that $C(A)$ is the norm closure of $S(A)$.

KEY WORDS AND PHRASES. *Commutative Banach algebra, semi-simple, socle, compactum, spectrum, carrier space, idempotent.*

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1. INTRODUCTION

Let A be a Banach algebra. For $x \in A$ let T_x denote the operator defined by $T_x(a) = xax$. The compactum of A is defined to be the set $\{x \in A : T_x \text{ is a compact operator on } A\}$. A Banach algebra A in which $A = C(A)$ is called a compact Banach algebra. Compact Banach algebras were first introduced by J. C. Alexander in [1]. The author, in [3], investigated the properties of the compactum in Banach Algebras. It was shown in [3] that if A is semi-simple and $S(A)$ denotes the socle of A , then $C(A)$ is non-zero if and only if $S(A)$ is non-zero, and in this case, $S(A) \subset C(A)$. Moreover, $C(A)$ is a closed set, therefore it contains the closure of $S(A)$. A problem of interest is to determine sufficient conditions on A which imply that $C(A)$ coincides with $\overline{S(A)}$. In [2], the author proved that for a primitive B^* algebra A , we have $C(A) = \overline{S(A)}$.

The purpose of this note is to prove that for a semi-simple commutative Banach algebra A , we have $C(A) = \overline{S(A)}$.

2. MAIN RESULT

To prove our theorem we use a result from [3] which states that if $x \in C(A)$ then the spectrum of $x(\sigma(x))$ is at most countable and 0 is its only possible accumulation point. Our terminology and notation is consistent with that of [4], and our algebras are over the field of complex numbers.

THEOREM: Let A be a semi-simple commutative Banach algebra. If $C(A)$ exists, then $C(A) = \overline{S(A)}$.

PROOF: We need to show that $C(A) \subset \overline{S(A)}$, as the other inclusion was already proven in [3].

Let Φ denote the space of multiplicative linear functionals in A , i.e. Φ is the carrier space of A .

Let $\chi \in C(A)$. We have, from general theory of commutative Banach algebras, $\sigma(\chi) = \{\hat{\chi}(\phi) : \phi \in \Phi\} \cup \{0\}$ where $\hat{\chi}$ is the continuous function on Φ defined by $\hat{\chi}(\phi) = \chi(\phi)$.

We claim that if ϕ is not an isolated point of Φ , then $\hat{\chi}(\phi) = 0$.

This is true because if $\hat{\chi}(\phi) \neq 0$ and $\{\phi_n\} \subset \Phi$ with $\lim_{n \rightarrow \infty} \phi_n = \phi$, then by the continuity of $\hat{\chi}$ we get $\hat{\chi}(\phi) = \lim_{n \rightarrow \infty} \hat{\chi}(\phi_n)$. But $\hat{\chi}(\phi_n)$ and $\hat{\chi}(\phi)$ belong to $\sigma(\chi)$. Therefore,

$\hat{\chi}(\phi)$ is a non-zero accumulation point of $\sigma(\chi)$ which is impossible since $\chi \in C(A)$.

Now since $\sigma(\chi)$ is countable, let $\{\phi_n\}$ be a sequence in Φ such that $\sigma(\chi) = \{\hat{\chi}(\phi_n) : n=1, 2, \dots\} \cup \{0\}$, where $\hat{\chi}(\phi_n) \neq 0$ for all n . Note that each ϕ_n is an isolated point of Φ .

Now, by Silov's idempotent theorem [4], for each $n = 1, 2, \dots$ there exists an idempotent $e_n \in A$ such that $\hat{e}_n(\phi) = 1$ and $\hat{e}_n(\phi) = 0$ if $\phi \neq \phi_n$. It is evident that e_n

is a minimal idempotent for each n .

For each m , let $x_n = \sum_{i=1}^n \hat{\chi}(\phi_i) e_i$. Then by the minimality of e_n , we have $x_n \in S(A)$.

Now, if $\{\phi_n\}$ is a finite set, then $\hat{\chi} = \hat{x}_n$ for some n and therefore $\hat{\chi} = \hat{x}_n \in S(A)$.

Otherwise, by the compactness of $\sigma(\chi)$ and the fact that 0 is the only possible accumulation point of $\sigma(\chi)$, we have $\lim_{n \rightarrow \infty} \hat{\chi}(\phi_n) = 0$.

Now, if $\phi \neq \phi_n$ for any n , then $\hat{\chi}(\phi) = 0$ and $\hat{e}_n(\phi) = 0$ for all n , thus $\hat{x}_n(\phi) = 0$ for all n . Moreover, $\hat{x}_n(\phi_m) = \hat{\chi}(\phi_m)$ if $n \geq m$ and 0 if $n < m$. Therefore, $\|\hat{x}_n - \hat{\chi}\| = \sup_{\phi \in \Phi} |\hat{x}_n(\phi) - \hat{\chi}(\phi)| = \sup_m |\hat{x}_n(\phi_m) - \hat{\chi}(\phi_m)|$, and we get $\lim_{n \rightarrow \infty} \|\hat{x}_n - \hat{\chi}\| = \lim_{n \rightarrow \infty} \sup_{m < n} |\hat{\chi}(\phi_m)| = 0$.

Therefore, $\lim_{n \rightarrow \infty} \hat{x}_n = \hat{\chi}$ and since the representation of A as an algebra of continuous functions on Φ is a homeomorphism, we get $\lim_{n \rightarrow \infty} x_n = \chi$.

Therefore, $\chi \in \overline{S(A)}$.

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