ANOTHER NOTE ON ALMOST CONTINUOUS MAPPINGS AND BAIRE SPACES

JINGCHENG TONG
Department of Mathematics
Wayne State University
Detroit, Michigan 48202 U.S.A.
(Received July 2, 1982)

ABSTRACT. The following result is proved:

Let Y be a second countable, infinite topological space with an ascending chain
of regular open sets. Then a topological space X is a Baire space if and only if
every mapping \( f: X \to Y \) is almost continuous on a dense subset of X.

It is another improvement of a theorem of Lin and Lin [2].

KEY WORDS AND PHRASES. Regular open set, almost continuous mapping, Baire space.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 54C10, 54F65.

1. INTRODUCTION.

In [1], the present author established a lemma by replacing Hausdorff space with
\( R_0 \)-space with an ascending chain of open sets. In this paper, a lemma is established
which has the same conclusion under independent conditions without any assumption on
separation, and it is used to give another improvement to a theorem of Lin and Lin [2].

2. MAIN RESULT.

An open set U in a topological space is a regular open set [3, p. 92] if
\( \text{Int}(\bar{U}) = U. \) Countably many regular open sets \( O_1, O_2, \ldots, O_n, \ldots \) is called an
ascending chain of regular open sets if \( O_1 \supset O_2 \supset \cdots \supset O_n \supset \cdots. \)

**LEMMA 1.** An infinite Hausdorff space has an ascending chain of regular open sets.

**PROOF.** By [4, Prob. 14, p. 147], we have a countably infinite subspace
\( \{y_1, y_2, \ldots, y_n, \ldots\} \) and disjoint open sets \( U_1, U_2, \ldots, U_n, \ldots \) such that \( y_n \in U_n \). Let
\( 0_n = \text{Int}(\bigcup_{i=1}^{n} U_i) \) \((n = 1, 2, \ldots)\). Then from [2, p. 92] we know that \( 0_n \) are regular open
sets. It is easily seen that \( y_n \notin 0_n. \) Since \( U_k \) are disjoint, \( y_n \notin \bigcup_{n-k}^{n} U_n \)
\((k = 1, 2, \ldots, n-1)\); hence, \( y_n \notin 0_{n-1}. \) Thus, \( 0_{n-1} \not\supset 0_n \) where \( (0_n, n = 1, 2, \ldots) \) is an
ascending chain of regular open sets.
The converse of Lemma 1 is not true.

**EXAMPLE 1.** Let \( D = \{d_1, d_2, \ldots, d_n, \ldots \} \) be an infinite set of distinct points. \( a, b, c \) are distinct points not in \( D \). Let \( X = \{a,b,c\} \cup D \) with topology \( \tau = \{N, \{a\} \cup N, \{a,b,c\} \cup N; N \text{ is a subset of } D\} \). Then \( 0_1 = \{d_1, d_2, \ldots, d_1\} \) \((i = 1, 2, \ldots)\) is an ascending chain of regular open sets. \( X \) is not \( T_0 \) since neither \( b \) nor \( c \) can be separated by open sets from the other. \( X \) is not \( R_0 \) since \( \{a\} = \{a,b,c\} \) does not belong to any \( \{a\} \) \( \tau N \).

In Example 1 of [1], \( X \) is the only regular open set. This shows that an \( R_0 \)-space with an ascending chain of open sets does not imply the existence of an ascending chain of regular open sets; thus, the two conditions are independent.

**LEMMA 2.** Let \( X \) be an infinite space with an ascending chain of regular open sets. Then \( X \) contains a countably infinite discrete subspace.

**Proof.** Let \( 0_1 (i = 1, 2, \ldots) \) be an ascending chain of regular open sets. Then \( V_n = 0_{n+1}/\overline{0}_n \) is a nonempty open set, otherwise \( 0_{n+1}/\overline{0}_n = \emptyset \) implies \( 0_{n+1} \subseteq \overline{0}_n \); hence, \( 0_{n+1} = \text{Int}(0_{n+1}) \subseteq \text{Int}(\overline{0}_n) = 0_n \), contradicting \( 0_n \nsubseteq 0_{n+1} \). Now we prove that \( \{V_n\} \) are disjoint. If \( m > n \), then \( V_m = 0_{m+1}/\overline{0}_m \cap V_n = \emptyset \), but \( 0_{n+1} \subseteq 0_m \); hence, \( V_m \cap \overline{0}_{n+1} = \emptyset \), \( V_n \subseteq 0_{n+1}/\overline{0}_n \subseteq 0_{n+1} \). Therefore, \( V_m \cap V_n = \emptyset \), \( \{V_n; n = 1, 2, \ldots\} \) are disjoint. Select a point \( y_n \in V_n \) for \( n = 1, 2, \ldots \); then, \( S = \{y_n; n = 1, 2, \ldots\} \) is a countably infinite discrete subspace.

Now, Theorems 2 and 3 in [2] can be written as follows:

**THEOREM 1.** Let \( Y \) be an infinite space with an ascending chain of regular open sets. If \( X \) is a topological space such that every mapping \( f: X \rightarrow Y \) is almost continuous on a dense subset of \( X \), then \( X \) is a Baire space.

**THEOREM 2.** Let \( Y \) be a second countable infinite space with an ascending chain of regular open sets. Then a topological space \( X \) is a Baire space if and only if every mapping \( f: X \rightarrow Y \) is almost continuous on a dense subset of \( X \).

**REMARK 1.** It is worth mentioning that, in Theorems 1 and 2, no separation property is required.

**REFERENCES**


Submit your manuscripts at http://www.hindawi.com