COMMUTATIVITY THEOREMS FOR RINGS AND
GROUPS WITH CONSTRAINTS ON COMMUTATORS

EVANGELOS PSOMOPOULOS
Department of Mathematics
University of Thessaloniki
Thessaloniki, Greece

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ABSTRACT. Let \( n > 1, m, t, s \) be any positive integers, and let \( R \) be an associative ring with identity. Suppose \( x^t[x^n,y] = [x,y^m]y^s \) for all \( x, y \) in \( R \). If, further, \( R \) is \( n \)-torsion free, then \( R \) is commutative. If \( n \)-torsion freeness of \( R \) is replaced by "\( m, n \) are relatively prime," then \( R \) is still commutative. Moreover, an example is given to show that the group theoretic analogue of this theorem is not true in general. However, it is true when \( t = s = 0 \) and \( m = n + 1 \).

KEY WORDS AND PHRASES. Commutative Rings, Torsion free Rings

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1. INTRODUCTION.

Throughout this note, \( R \) will be an associative ring with identity, \( Z \) the center \( R \), \( N \) the set of all nilpotent elements of \( R \), and \( C(R) \) the commutator ideal of \( R \). We set \( [x,y] = xy - yx \).

Our objective is to prove the following

THEOREM 1. Let \( n (> 1), m \) be positive integers and let \( t, s \) be any non-negative integers. Let \( R \) be an associative ring with identity. Suppose \( x^t[x^n,y] = [x,y^m]y^s \) for all \( x, y \) in \( R \). If, further, \( R \) is \( n \)-torsion free, then \( R \) is commutative.

In preparation for the proof of this theorem, we first establish the following lemmas.

LEMMA 1. Let \( R \) be a ring with \( 1, k \) any positive integer, and let \( x, y \) be in \( R \).

(i) If \( [x,[x,y]] = 0 \), then \( [x^k,y] = kx^{k-1}[x,y] \).
(ii) If \( x^ky = 0 = (x+1)^ky \), then \( y = 0 \).
(iii) If \( (m,n) = 1 \) and \( [x^n,y] = [x^m,y] = 0 \), for all \( x \) in \( R \), then \( [x,y] = 0 \).

This lemma is very well-known.
LEMMA 2. Under the hypotheses of the above theorem, every nilpotent element of $R$ is central.

PROOF. It is a triviality to prove that hypothesis
$$x^t[x^n, y] = [x, y^m]^y^s$$ for all $x, y$ in $R$ (1.1)
implies
$$x^t'[x^n, y] = [x, y^m]^y^{s'}$$ for all $x, y$ in $R$ $t' = nt + t$, $s' = 2s$ (1.2)

Let $x \in N$; then there exists a positive integer $p$, such that
$$a^k \in Z$$ for all $k \geq p$, $p$ minimal. (1.3)

Suppose $p > 1$. In (1.1), replace $x$ by $a^{p-1}$ to get
$$(a^{p-1})^t[(a^{p-1})^n, y] = [a^{p-1}, y^m]^y^s$$
which implies, in view of (1.3),
$$[a^{p-1}, y^m]^y^s = 0$$ (1.4)

Now, in (1.1) replace $x$ by $1 + a^{p-1}$, to obtain
$$(1 + a^{p-1})^t[(1 + a^{p-1})^n, y] = [a^{p-1}, y^m]^y^s.$$ (1.5)

In view of (1.4), and the fact that $1 + a^{p-1}$ is invertible, the last equation implies
$$[(1 + a^{p-1})^n, y] = 0.$$ (1.5)

Combining (1.5) and (1.3), we see that
$$0 = [(1 + a^{p-1})^n, y] = [1 + na^{p-1}, y] = n[a^{p-1}, y].$$

Since $R$ is $n$-torsion free, the last identity implies $[a^{p-1}, y] = 0$, for all $y$ in $R$, which contradicts the minimality of $p$. This contradiction shows that $p = 1$. Therefore, $N \subset Z$.

Now, observe that by [1, Theorem 1], $C(R)$ is a nil ideal, since $x = e_{22}$ and $y = e_{22} - e_{22}$ fail to satisfy (1.1). Hence in view of Lemma 2, we obtain
$$C(R) \subseteq Z$$ (1.6)

PROOF OF THEOREM 1. In (1.1), replace $x$ by $2x$ to get
$$2^n + 2x^t[x^n, y] = 2[x, y^m]^y^s.$$ (1.7)

Combining the last identity with (1.1), we obtain
$$2^n + 2^n t[x, y^m]^y^s = 2[x, y^m]^y^s.$$ (1.7)

In view of (1.6) and Lemma 1, (1.7) yields
$$2^n + t[m, y^m+s-1[x, y] = 2m[y^m+s-1[x, y]]$$
$$(2^n + t)([m, y^m+s-1[x, y]] = 0.$$ (1.8)

Then, if $k = (2^n + t - 2)m(1+s)$, $[x, y^k] = ky^k-1[x, y] = 0$. Therefore,
$$x^k \in Z$$ for all $x \in R$; $k = (2^n + t - 2)m(1+s).$ (1.8)

Next, by (1.1) we obtain
$$x^t[x^n, y] = my^m+s-1[x, y].$$

Replace $y$ by $y^m$ in the above equation to get
$$x^t[x^n, y^m] = my^m+s-1[x, y^m]$$
$$m^n[x^n, y^m] = my^m+s-1[x, y^m].$$ (1.9)

Combining the last identity with (1.1) and (1.6), we obtain
$$m^n[x, y^m] = my^m+s-1(1-y^m)(m+s-1) = 0.$$ (1.9)
Multiply (1.9) by $y^{(m-1)(m+s-1)}$ to obtain
\[ m[x,y]y^{m+s-1}(y^{(m-1)(m+s-1)} - y^{2(m-1)(m+s-1)}) = 0. \] (1.10)

Adding together (1.9) and (1.10), we see that
\[ m[x,y]y^{m+s-1}(1-y^{2(m-1)(m+s-1)}) = 0. \]

Continue this process \( k \) times (\( k \) being as in (1.8)) to obtain
\[ m[x,y]y^{m+s-1}(1-y^{k(m-1)(m+s-1)}) = 0. \] (1.11)

It is well known that \( R \) is isomorphic to a subdirect sum of subdirectly irreducible rings \( R_i \) (i.e.\). Each \( R_i \) satisfies (1.2), (1.6), (1.8), and (1.11), but \( R_i \) is not necessarily \( n \)-torsion free.

We consider the ring \( R_i \) (i.e.\). Let \( S \) be the intersection of all non-zero ideals of \( R_i \). Then, it can be easily verified
\[ Sd = 0, \text{ for all central zero divisors } d \] (1.12)

If \( a \) is any zero divisor of \( R_i \), then
\[ m[x,a]^{m+s-1}(1-a^{k(m-1)(m+s-1)}) = 0. \]
Thus,
\[ m[x,a]a^{m+s-1} = 0 \] (1.13)
For if \( m[x,a]a^{m+s-1} \neq 0 \), then \( a^{k(m-1)(m+s-1)} \) will be a central (see (1.8)) zero divisor and by (1.12), \( 0 = S(1-a^{k(m-1)(m+s-1)}) = S \), a contradiction. Combining (1.2) and (1.13), we see that
\[ xt'[x^n,a] = [x,a]^{m+s-1} = m[x,a]^{m-1} + s'. \]
Hence by Lemma 1,
\[ n^2x^{n^2} + t'^{-1}[x,a] = xt'[x^n,a] = 0. \]
Replacing \( x \) by \( x+1 \) in the last identity and using Lemma 1, we obtain
\[ n^2[x,a] = 0, \text{ which yields } [x^n,a] = n^2x^{n^2-1}[x,a] = 0. \] Therefore,
\[ [x^n,a] = 0, \text{ for all } x \text{ in } R_i, \text{ and all zero divisors } a \text{ of } R_i. \] (1.14)

Next, let \( c \) be any central element of \( R_i \). In (1.1), replace \( x \) by \( cx \) to get
\[ c^{n+t}x^{t}[x^n, y] = c[x,y]^{m}y^{s} = cx^{t}[x^n, y] \]
\[ (c^{n+t}-c)x^{t}[x^n, y] = 0. \]
Apply once more Lemma 1 to obtain
\[ n(c^{n+t}-c)x^{n+t-1}[x,y] = 0. \]
If we replace \( x \) by \( x+1 \), and apply Lemma 1, we finally get
\[ n(c^{n+t}-c)[x,y] = 0, \text{ which implies } \]
\[ (c^{n+t}-c)[x^n, y] = 0, \text{ for all } x,y \in R_i, \text{ and any central element } c \text{ of } R_i. \] (1.15)
In particular,
\[ (y^{k(n+t)}-y^k)[x^n, y] = 0 \text{ for all } x,y \in R_i. \] (1.16)
Now, let \( y \in R_1 \). If \( [y, x^{n^2}] = 0 \), then clearly \( [y^q, x^{n^2}] = 0 \) for all positive integers \( q \). If \( [y, x^{n^2}] \neq 0 \), then \( [y, x^n] \neq 0 \). For \( [x^n, y] = 0 \) implies \( [y, x^{n^2}] = 0 \), a contradiction. Since \( [x^n, y] \neq 0 \), (1.16) implies that \( y^{k(n+t)} \cdot y^k \) is a zero divisor. Therefore, \( y^{k(n+t-1)+1} \cdot y \) is also a zero divisor. Hence, (1.14) implies

\[
[y^p, x^{n^2}] = 0 \quad \text{for all} \quad y \in R_1; \quad p = k(n+t)+1 \tag{1.17}
\]

Since each \( R_i \) (i \( \in \gamma \)) satisfies (1.17), the original ring \( R \) also satisfies (1.17). But \( R \) is n-torsion free. Thus, combining (1.17) and Lemma 1, we finally obtain

\[
[y^p, y, x] = 0, \quad \text{for all} \quad y \in R,
\]

which implies commutativity of \( R \) by Herstein's theorem [3].

2. If we replace, in Theorem 1, hypothesis "\( R \) is n-torsion free" by the condition "\( n \) and \( m \) are relatively prime," the ring \( R \) is still commutative.

THEOREM 2. Let \( n, m \) be relatively prime positive integers, and let \( t, s \) be any non-negative integers. Suppose \( R \) is an associative ring with identity satisfying \( x^t[x^n, y] = x^m y^s \) for all \( x, y \) in \( R \). Then \( R \) is commutative.

PROOF. Here, without loss of generality, we assume that \( R \) is subdirectly irreducible.

Let \( a \in \mathbb{N} \). Following the same argument as in Theorem 1, we prove (see (1.5)) that \( n[a, x^n] = 0 \) for all \( y \in R \); similarly, we can prove that \( m[a, x^m] = 0 \) for all \( y \in R \). Since \( (m,n) = 1 \), we obtain

\[
C(R) \subseteq N \subseteq Z. \tag{2.1}
\]

Note that the proof of (1.8) also works in the present situation, so that there exists \( k \) for which

\[
x^k \in Z \quad \text{for all} \quad x \in R. \tag{2.2}
\]

Furthermore, as in the proof of Theorem 1 we obtain \( [x^{n^2}, a] = 0 \) for all \( x \in R \) and all zero divisors \( a \) (see (1.14)); similarly \( [x^{m^2}, a] = 0 \). Thus, the last part of Lemma 1 yields

\[
[x, a] = 0 \quad \text{for all} \quad x \in R \quad \text{and all zero divisors} \quad a. \tag{2.3}
\]

As we observed in the paragraph following (1.14), we have \( n(c^{n+t})[x, y] = 0 \) for all \( x, y \in R \) and all \( c \in Z \); and a variation of the argument yields \( m(c^{n+t})[x, y] = 0 \) as well. Thus

\[
(c^{n+t})[x, y] = 0 \quad \text{for all} \quad x, y \in R \quad \text{and all} \quad c \in Z. \tag{2.4}
\]

Using (2.2) to substitute \( y^k \) for \( c \), we complete the proof by arguing as in the previous proof that \( y^{k(n+t-1)+1} \cdot y \in Z \) for all \( y \in R \). Hence, \( R \) is commutative by Herstein's theorem [3].
3. A close look at the symmetric group $S_3$ with $t=s=6$, $n=7$ and $m=1$ shows that $S_3$ satisfies the identity $x^t[x^n,y] = [x,y]^n y^5$. But, as it is well known, $S_3$ is not abelian. Hence, Theorem 2 is not true for groups in general. However, we prove the following:

**Theorem 3.** Let $G$ be a multiplicative group, $n$ an arbitrary positive integer, and suppose $[x^n,y] = [x,y^{n+1}]$ for all $x,y$ in $G$. Then $G$ is abelian.

**Proof:** In hypothesis, replace $x$ by $xy$ to obtain

$$[(xy)^n, y] = [xy, y^{n+1}].$$

(3.1)

A direct calculation shows that $[xy, y^{n+1}] = [x, y^{n+1}]$. Combining this with hypothesis and (3.1) we see that $[(xy)^n, y] = [x^n, y]$. Replace $y$ by $x^{-1}y$, in the last equation to get

$$[y^n, x^{-1}y] = [x^n, x^{-1}y].$$

(3.2)

A direct calculation shows that $[y^n, x^{-1}y] = [x^n, x^{-1}]$, and $[y^n, x^{-1}] = x^{-1}[x^n, y]x$. Thus (3.2) yields $[y^n, x^{-1}] = x^{-1}[x^n, y]x$, which yields

$$x[y^n, x^{-1}] = [x^n, y]x = [x, y^{n+1}]x.$$

Hence,

$$xy^{n+1}x^{-1}y^{-n-1}x = xy^n x^{-1}y^{-n}x$$

and after cancellations $yx^{-1}y^{-1} = x^{-1}$, which implies $xy = yx$. Hence, $G$ is abelian.

4. We conclude with the following

**Remark.** As a corollary to Theorem 1, with $t=s=0$ and $m=n$, we obtain the following result of Bell [2, Theorem 5]:

**Corollary.** Let $R$ be a ring with $1$ and $n>1$ a fixed positive integer. If $R$ is $n$-torsion free and $R$ satisfies the identity $x^n y - y x^n = x y^n - y^n x$, then $R$ is commutative.

Also, Theorem 1 generalizes a result of E. Psomopoulos, H. Tominaga, and A. Yaqub [4, Theorem 2].

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**References**
