ON CERTAIN REGULAR GRAPHS OF GIRTH 5

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ABSTRACT. Let \( f(v, 5) \) be the number of vertices of a \((v, 5)\)-cage \((v \geq 3)\). We give an upper bound for \( f(v, 5) \) which is considerably better than the previously known upper bounds. In particular, when \( v = 7 \), it coincides with the well-known Hoffman-Singleton graph.

KEY WORDS AND PHRASES. Regular graph, cage, latin square.

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A graph is said to be regular of valency \( v \) if each of its vertices has valency \( v \). A regular graph of valency \( v \) and girth \( g \) with the least possible number of vertices is called a \((v, g)\)-cage. The number of vertices of a \((v, g)\)-cage is denoted by \( f(v, g) \). The existence of \((v, g)\)-cages was proved by Erdős and Sachs [4]. In this paper, we consider only regular graphs of girth 5. It is easy to see that \( f(v, 5) \geq v^2 + 1 \). Also, it is known that \( f(3, 5) = 10 \) [1], \( f(4, 5) = 19 \) [8], \( f(5, 5) = 30 \) [10], \( f(6, 5) = 40 \) [6], and \( f(7, 5) = 50 \) [5]. For \( v > 5 \), Brown [2] has shown that \( f(v, 5) \leq 2(2v-1)(v-2) \). In [10], Wegner has shown that \( f(v, 5) \leq 2v(v-1) \) for primes \( v \geq 3 \). In Theorem 1b of [7], Parsons implicitly proved that \( f(v, 5) \leq 2v^3 - 3v + 1 \) when \( v \) is odd.

Notation. If two vertices \( x \) and \( y \) in a graph are adjacent, we write \( x \sim y \).

We now give a better bound for \( f(v, 5) \).

Theorem 1. Let \( v \geq 7 \) be an integer such that \( v-2 \) is a prime power. Then the following statements hold:

(a) \( f(v, 5) \leq 2(v-2)^2 \).

(b) If \( n \) is an integer such that \( 3 \leq n \leq v \), then \( f(n, 5) \leq 2(v-2)(n-2) \).
We use the same notations as in [3, p. 169]. If \( R = p^n \) is a prime power, then a set of \( R-1 \) mutually orthogonal latin squares of order \( R \) can be constructed. In fact, let the elements of the Galois field \( GF[R] \) be denoted by \( u_0 = 0, u_1 = 1, u_2 = x, u_3 = x^2, \ldots, u_{R-1} = x^{R-2} \), where \( x \) is a generating element of the multiplicative group of \( GF[R] \) and \( x^{R-1} = 1 \). Then a complete set of mutually orthogonal latin squares \( L_1, L_2, \ldots, L_{R-1} \) can be obtained as follows. \( u_i + u_j \) is the entry in the \( i^{th} \) row and \( j^{th} \) column of square \( L_1 \).

As in [3], we use the symbol \( u_i + u_j \) within a square to stand for the integer \( k \), where \( u_i + u_j = u_k \) (\( i, j, k = 0, 1, \ldots, R-1 \)). The elements of the \( 0^{th} \) row of each \( L_i \) (\( i = 1, 2, \ldots, R-1 \)) are identical and the remaining rows of \( L_i \) (\( i = 2, 3, \ldots, R-1 \)) are obtained by permuting cyclically the remaining rows of \( L_1 \). We let

\[
L_0 = \begin{bmatrix}
0 & 1 & \ldots & R-1 \\
0 & 1 & \ldots & R-1 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \ldots & R-1
\end{bmatrix}
\]

to obtain \( R \) mutually orthogonal squares.

**Proof of Theorem 1.** (a) Let \( R = v-2 \) (a prime power). We give an explicit construction of a regular graph \( G \) ofirth 5 and valency \( v \), having \( 2R^2 \) vertices. In fact, let the vertices of \( G \) be arranged as in Figure 1.
We say that the vertices \( \{x_k^0, x_k^1, \ldots, x_k^{R-1}\} \) are in set \( X_k \). Similarly we define set \( Y_i \) (\( i, k = 0, 1, \ldots, R-1 \)). We join the set of vertices \( Y_i \) to the set \( X_k \) (for all \( i, k \)) according to the following rule. If \( n \) is an integer in the \( i \)th row and \( j \)th column of \( L_k \), then the vertex \( y_n^i \) is adjacent to vertex \( x_n^j \) (\( i, j, k = 0, 1, \ldots, R-1 \)). Since the squares \( \{L_0, L_1, \ldots, L_{R-1}\} \) are mutually orthogonal, it is readily seen that the graph has girth 5 and valency \( R (= v-2) \). To increase this valency to \( v \), it suffices to join each vertex \( x_n^j \) (resp. \( y_n^i \)) to two other vertices in the same set \( X_k \) (resp. \( Y_i \)) (\( i, k = 0, 1, \ldots, R-1 \)) in such a way that the girth of the graph \( G \) remains unchanged.

For any integer \( n \) (\( n = 0, 1, \ldots, R-1 \)), we let \( \mathbb{A}_n \) be the set of pairs of integers \( \{(u_{n+u}, u_{n+u}), (u_{n+u}, u_{n+u}), \ldots, (u_{n+u}, u_{n+u})\} \) in \( L_1 \). The first (resp. second) terms of the pairs are integers ranging from \( 0 \) to \( R-1 \). Each row of every square contains every integer from \( 0 \) to \( R-1 \). Suppose a pair \([s, t] \) in \( \mathbb{A}_n \) appears in the \( i \)th row and the \( s_1 \) and \( t_1 \) columns of \( L_k \). Then we associate \( s \) and \( t \) with the two vertices \( x_n^s \) and \( x_n^t \) in set \( X_k \) and also with the two vertices \( y_n^s \) and \( y_n^t \) in set \( Y_i \) (\( i, k = 0, 1, \ldots, R-1 \)).

We define \( \mathbb{A}_n \) to be the set of all pairs of vertices so associated with the \( \cdots \). The joining of a pair of vertices belong to \( \mathbb{A}_n \), we call an \( \mathbb{A}_n \)-join.

We know from the construction of squares \( L_k \) (\( k = 2, 3, \ldots, R-1 \)) from \( L_1 \), that a pair of integers \([s, t] \) appears in some row and in columns \( s_1 \) and \( t_1 \) of \( L_k \) if and only if the same pair appears in some row and in columns \( s_1 \) and \( t_1 \) of \( L_1 \). Thus in what follows, we need only look at squares \( L_1 \).

Lemma 2. Let \([s, t] \in \mathbb{A}_n \) and \([v, w] \in \mathbb{A}_m \). Then \([v, w] \in \mathbb{A}_n \) if and only if \( v = s \) and \( w = t \) or \( v = t \) and \( w = s \). Equivalently, \( \mathbb{A}_n \cap \mathbb{A}_m = \emptyset \) if and only if \( n \neq m \) (\( m, n = 1, 2, \ldots, R-1 \)).

Proof. Suppose a pair of integers \([s, t] \) appears in row \( i \) and columns \( s_1 \) and \( t_1 \) and also in row \( I \) and column \( S \) and \( T \) of square \( L_1 \). Then another pair, say \([v, w] \), of integers appears in some row \( i' \) and columns \( s_1 \) and \( t_1 \) if and only if \([v, w] \) also appears in some row, say \( I' \) and columns \( S \) and \( T \).

In fact, let \( z \) be the integer in column \( T \) and in the same row \( I' \) as the \( v \) which appears in column \( S \) (see Figure 2).
Then

\[\begin{align*}
  u_s &= u_i + u_{s_1} = u_I + u_S \\
  u_t &= u_i + u_{t_1} = u_I + u_T \\
  u_w &= u_v + u_{t_1} - u_{s_1} \\
  u_z &= u_v + u_T - u_S \\
  &= u_v + (u_i + u_{t_1} - u_{s_1}) - (u_i + u_{s_1} - u_I) \\
  &= u_v + u_T - u_S = u_w.
\end{align*}\]

It follows that if a pair of integers \([s,t]\) is in set \(\Xi_n\), then it does not belong to set \(\Xi_m\) for \(n \neq m\). This completes the proof of the lemma.

Remark 1. If we use a collection of sets \(\{\Xi_n : n \in N\}\) to join together the vertices of each set \(X_k (k = 0, 1, \ldots, R-1)\) and a different collection of sets \(\{\Xi_m : m \in M\}\), where \(M \cap N = \{\emptyset\}\) to join together the vertices of each set \(Y_i (i = 0, 1, \ldots, R-1)\), then the girth of the graph \(G\) remains five. In fact, suppose \(x^k_{s_1} \sim x^k_{t_1}\), where \([x^k_{s_1}, x^k_{t_1}] \in A_n\), then by construction of \(G\), \(y^i_s \sim x^k_{s_1}\) and \(y^i_t \sim x^k_{t_1}\). But by the above lemma, \([y^i_s, y^i_t] \notin A_n\) and \(y^i_s \sim y^i_t\) under an \(A_m\)-join \(m \neq n\). That is \([y^i_s, y^i_t] \notin A_m\), \(m \neq n\).

We divide the remaining proof of Theorem 1(a) into three cases.

Case 1. Assume \(R = 2^r\) (\(r \geq 3\)). The vertices of \(X_k (k = 0, 1, \ldots, R-1)\) are joined together to form \(2^{r-3}\) mutually disjoint 8-gons by using the pattern \(A_1 A_2 A_3 A_4 A_5 A_6 A_7\). Explicitly,
\[ x^k_{u_0} \sim x^k_{u_0+u_1} \sim x^k_{u_0+u_1+u_2} \sim x^k_{u_0+u_2} \sim \]

\[ x^k_{u_0+u_2+u_3} \sim x^k_{u_0+u_1+u_2+u_3} \sim x^k_{u_0+u_1+u_3} \sim x^k_{u_0+u_3} \sim x^k_{u_0}. \]

It is easy to see that these eight vertices are distinct and they form an 8-ron and, use the same pattern, \( A_1A_2A_1A_3A_2A_1A_3 \), starting with \( x^k_{u_0} \) to get a second 8-ron, the vertices of which are clearly distinct from those of the first. If \( r > 4 \), we repeat this last step until we have \( 2^{r-3} \) 8-rons. Similarly, the vertices of \( Y_i \) \((i = 0, 1, \ldots, R-1)\) are joined together to form \( 2^{r-3} \) mutually disjoint 8-rons by using the pattern \( A_4A_5A_6A_4A_5A_6 \).

\[ y^i_{u_0} \sim y^i_{u_0+u_4} \sim y^i_{u_0+u_4+u_5} \sim y^i_{u_0+u_5} \sim \]

\[ y^i_{u_0+u_5+u_6} \sim y^i_{u_0+u_4+u_5+u_6} \sim y^i_{u_0+u_4+u_6} \sim y^i_{u_0+u_6} \sim y^i_{u_0}. \]

Thus \( G \) has valency \( v \). It remains to show that \( G \) has girth 5. Since we use \( A_1A_2A_1A_3A_1A_2A_1A_3 \) -joins in \( X_k \) and \( A_4A_5A_6A_4A_5A_6 \) -joins in \( Y_i \), it follows from Lemma 2 and Remark 1 that any pair of vertices in \( X_k \) which we join are not adjacent to any pair of vertices joined in \( Y_i \). (i, k = 0, 1, \ldots, \( R-1 \)). This statement is true for \( X_0 \) and \( Y_0 \). Because of the construction of \( L_0 \), hence the graph \( G \) does not contain any 4-rons. Therefore \( G \) has girth 5. This completes the proof of Case 1.

Example. For \( r = 2^3 = 8 \), we have

\[ L_1 = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 4 & 7 & 2 & 6 & 5 & 3 \\
2 & 4 & 0 & 5 & 1 & 3 & 7 & 6 \\
3 & 7 & 5 & 6 & 0 & 2 & 4 & 1 \\
4 & 2 & 1 & 6 & 0 & 7 & 3 & 5 \\
5 & 6 & 3 & 2 & 7 & 0 & 1 & 4 \\
6 & 5 & 7 & 4 & 3 & 1 & 0 & 2 \\
7 & 3 & 6 & 1 & 5 & 4 & 2 & 0
\end{bmatrix} \]
\[ x_0^k \sim x_1^k \sim x_4^k \sim x_2^k \sim x_5^k \sim x_7^k \sim x_3^k \sim (x_0^k) \]

and

\[ y_0^i \sim y_4^i \sim y_5^i \sim y_7^i \sim y_2^i \sim y_3^i \sim y_6^i \sim (y_0^i) \]

where \( i, j = 0, 1, \ldots, 7 \).

**Case 2.** Assume \( P = 3^r \) (\( r \geq 2 \)). The vertices of \( X_k \) are joined together to form \( 3^{r-2} \) mutually disjoint 9-gons by using the pattern \( A_1 A_2 A_1 A_2 A_1 A_2 \). Explicitly,

\[ x_0^k \sim x_0^k + u_1 \sim x_0^k + u_1 + u_2 \sim x_0^k + u_1 + u_2 + u_3 \sim \]

\[ x_0^k + u_2 \sim x_0^k + u_1 + u_2 \sim x_0^k + u_1 + u_2 + u_2 \sim \]

\[ x_0^k + u_1 + u_2 + u_2 \sim (x_0^k). \]

Repeating this pattern, we form \( 3^{r-2} \) mutually disjoint 9-gons from \( X_k \) (\( k = 0, 1, \ldots, R-1 \)). Similarly, the vertices of \( Y_1 \) are joined together to form \( 3^{r-2} \) mutually disjoint 9-gons by using the pattern \( A_3 A_4 A_3 A_4 A_3 A_4 \). Thus \( G \) has valency \( v \). It follows from Lemma 2 and Remark 1 that \( G \) has girth 5.

**Case 3.** Assume \( R = p^r \) where \( p \) is a prime and \( r \geq 1 \). Join the vertices of \( X_k \) using the pattern \( A_1 A_{1 \ldots A_1} \) to produce \( r \) mutually disjoint \( p \)-gons. Explicitly, the first \( p \)-gon is

\[ x_0^k \sim x_0^k + u_1 \sim x_0^k + u_1 + u_1 \sim \ldots \sim x_0^k + u_1 + u_1 + \ldots + u_1 \sim (x_0^k). \]

Similarly, we join the vertices of \( Y_1 \) using the pattern \( A_2 A_2 A_2 \) to get \( r \) mutually disjoint \( p \)-gons (\( i, j = 0, 1, \ldots, R-1 \)). Therefore \( G \) has valency \( v \).

It follows from Lemma 2 and Remark 1 that \( G \) has girth 5. This completes the proof of (a).

(b). Let \( G \) be the graph constructed as in (a). The subgraph of \( G \) induced by \( X_0, X_1, \ldots, X_{n-3}, Y_0, Y_1, \ldots, \) and \( Y_{n-3} \) clearly has girth 5 and valency \( n \) with order \( 2(v-2)(n-2) \). This completes the proof of Theorem 1.

**Remark 2.** Let \( v \) be an integer \( \geq 3 \). Since there always exists a prime power \( R \) such that \( R \geq v \), it follows that Theorem 1 gives an upper bound for \( f(v,5) \) for any \( v \geq 3 \).
Remark 3. For $v = 7$, Theorem 1 is identical with the construction of the Hoffman-Singleton graph given in [1] and [9], and $f(7,5) = 50$. The upper bound for $f(v,5)$ given in Theorem 1 is better than the other bound mentioned previously. For example, we have $f(9,5) \leq 98$ and $f(8,5) \leq 84$.

Remark 4. For $p = v - 1$ a prime number, a set of mutually orthogonal latin squares $L_1, L_2, \ldots, L_{p-1}$ is more easily obtained by a simple rotation of the elements in the rows of $L_0$, which is the same as before. Explicitly

$$L_1 = \begin{bmatrix}
0 & 1 & 2 & 3 & \ldots & p-1 \\
p-1 & 0 & 1 & 2 & \ldots & p-2 \\
p-2 & p-1 & 0 & 1 & \ldots & p-3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \ldots & \ldots & \ldots & \ldots & 0
\end{bmatrix}$$

In general, in $L_k$ ($k = 2, 3, \ldots, p-1$), if $k$ appears in row $i$ and column $j$, then 0 is in row $i + 1$ and column $j$. This simplifies the construction of the graph.

REFERENCES


