RESEARCH NOTES

BOUNDED SETS IN FAST COMPLETE INDUCTIVE LIMITS

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(Received April 26, 1984)

ABSTRACT. Let \( E_1 \subseteq E_2 \subseteq \ldots \) be a sequence of locally convex spaces with all identity maps: \( E_n \to E_{n+1} \) continuous and \( E = \text{indlim}\ E_n \) fast complete. Then each set bounded in \( E \) is also bounded in some \( E_n \) iff for any Banach disk \( B \) bounded in \( E \) and \( n \in \mathbb{N} \), the closure of \( B \cap E_n \) in \( B \) is bounded in some \( E_m \). This holds, in particular, if all spaces \( E_n \) are webbed.

KEY WORDS AND PHRASES. Inductive limit of locally convex spaces, fast complete space, webbed space, bounded set.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 46A12, Secondary 46A07

Throughout the paper \( E_1 \subseteq E_2 \subseteq \ldots \) is a sequence of Hausdorff locally convex spaces with continuous identity maps: \( E_n \to E_{n+1} \), and \( E = \text{indlim}\ E_n \) also Hausdorff.

For an absolutely convex set \( A \) in a locally convex space \( X \) we denote by \( X_A \) the linear hull of \( A \) equipped with the topology generated by \( \{ \lambda A; \lambda > 0 \} \). If \( X_A \) is a Banach space, \( A \) is called a Banach disk, see [1]. The space \( X \) is fast complete if each set bounded in \( X \) is contained in a Banach disk which is bounded in \( X \). Every sequentially complete locally convex space is fast complete.

If \( B \subseteq C \subseteq X \), the closure of \( B \) in \( C \) is denoted by \( \overline{B}^C \). For brevity we denote by \( \text{DS} \), resp. \( \text{DST} \), the following property: each set bounded in \( E \) is contained, resp. bounded, in some \( E_n \).

We use the notion of webbed spaces, see [1] or [2], to derive our first criterion for \( \text{DST} \).

THEOREM 1. If all spaces \( E_n \) are webbed and \( E \) is fast complete, then \( \text{DST} \) holds.

PROOF. Let \( A \subseteq E \) be bounded. Then \( A \) is contained in a Banach disk \( B \) bounded in \( E \). The space \( E_B \) is Banach and the identity map \( \text{id}: E_B \to \text{indlim}\ E_n \) is continuous. By the Corollary IV.6.5 of [1], there exists \( n \in \mathbb{N} \) such that \( E_B \subseteq E_n \) and \( \text{id}: E_B \to E_n \) is continuous. Since \( A \) is bounded in \( E_n \), it is bounded in \( E_n \).

REMARK. The same result is proved in [3] for strictly webbed spaces.

It is evident that if all spaces \( E_n \) are fast complete then \( \text{DST} \) implies fast completeness of \( E \). In particular, since every Fréchet space is webbed and fast complete, we have: If all spaces \( E_n \) are Fréchet then \( \text{DST} \) holds iff \( E \) is fast complete.
PROPOSITION. Let \( B \) be a Banach disk bounded in \( E \). Then \( B = \overline{B \cap E_m^B} \) for some \( m \in \mathbb{N} \).

PROOF. Put \( B_n = \overline{B \cap E_n^B} \) and \( F_n = E_n^B \), \( n \in \mathbb{N} \). The set \( B_n \) is closed in the Banach space \( E_B \), hence \( F_n \) is Banach and as such it is also webbed. Since each id: \( F_n \to E_B \) is continuous, the map id: \( \text{indlim } F_n \to E_B \) is continuous too and its graph is fast sequentially closed. Hence the inverse mapping id: \( E_B \to \text{indlim } F_n \) has also fast sequentially closed graph and, by Corollary IV.6.5 of [1], \( E_B \subset F_m^B \) for some \( m \).

Assume there exists \( b \in B \setminus B_m^B \) and put \( \beta = \inf\{\alpha > 0; b \in \alpha B_m^B\} \). Evidently \( b \in \beta B_m^B \). Hence \( \beta > 1 \). There exists a sequence \( \{b_k\} \subset B \cap E_m^B \) such that \( b_k \to \beta^{-1} b \) is the topology of \( E_B \). Take \( \gamma \in (1, \beta) \). Then \( \|\beta^{-1} b\| < \|\gamma^{-1} b\| \) and \( \|b_k\| < \|\gamma^{-1} b\| \) for sufficiently large \( k \)'s. For the same \( k \)'s, we have \( \|\gamma b_k\| < \|b\| \), which means \( \gamma b_k \in B \). Further \( \gamma b_k \in E_m \) and \( \gamma b_k \to \gamma \beta^{-1} b \in B_m \) in the topology of \( E_B \), i.e. \( b \in \gamma^{-1} \beta B_m^B \), a contradiction.

THEOREM 2. Let \( E \) be fast complete. Then DS, resp. DST, holds iff for any Banach disk \( B \) bounded in \( E \) and any \( n \in \mathbb{N} \), \( B \cap E_n^B \) is contained, resp. bounded, in some \( E_m \).

PROOF. "If" part is evident. For the "only if", take a set \( A \) bounded in \( E \), then \( A \) is contained in a Banach disk \( B \subset E \). By the Prop., there exists \( n \in \mathbb{N} \) such that \( B = \overline{B \cap E_n^B} \) which is contained, resp. bounded, in some \( E_m \).

EXAMPLE. Let FC stand for the property:

Each set bounded in \( E_n \) is contained in a bounded Banach disk in \( E \).

And let \( P_1 \), resp. \( P_2 \), stand for:

For each bounded Banach disk \( B \) in \( E \) and \( n \in \mathbb{N} \), the set \( \overline{B \cap E_n^B} \) is contained, resp. bounded, in some \( E_m \).

It follows from Theorem 2 that: \( A_1 \equiv E \) fast complete \( \Rightarrow\) DST \& FC, \( A_1 \equiv E \) fast complete \( \Rightarrow\) DS \& FC. The last implication cannot be reversed. To show that, take a Banach space \( X \), \( \dim X = +\infty \), denote by \( L \) its underlying vector space, and choose a subspace \( M \subset L \) which is dense in \( X \). Let \( Y \) be \( L \) equipped with the finest locally convex topology, \( V = \bigcup\{L^n \times M^n; n \in \mathbb{N}\} \), \( X_n = X^n \times Y^n \), and \( E_n \) be the vector space \( V \) with the topology inherited from \( X_n, n \in \mathbb{N} \).

The property DS holds, since the underlying vector spaces of all \( E_n \) are the same. We show that each \( E_n \) is quasi-complete. Hence it is fast complete, and FC trivially holds.

Let \( A \subset E_n \) be bounded. Then \( A \subset \bigcap \{A_k; k \in \mathbb{N}\} \), where \( A_k \) is bounded, closed, and absolutely convex in \( X \) for \( k \leq n \), and in \( Y \) for \( k > n \). Any set bounded in \( Y \) is contained and bounded in a finite dimensional subspace of \( Y \). Hence each \( A_k \), \( k \in \mathbb{N} \), is complete and \( A \) is contained in the complete set \( \bigcap \{A_k; k \in \mathbb{N}\} \).

The space \( E = \text{indlim } E_n \), which equals \( V \) with the topology inherited from \( X^n \), is not fast complete. Assume the contrary. If \( B \) is the closed unit ball in \( X \), then \( B_0 = B_0^N \cap V \) is bounded in \( E \) and a fortiori contained in a bounded Banach disk \( D \) in \( E \).

Take \( x_0 \in B \setminus M \), choose a sequence \( \{x_k\} \subset B \cap M \) such that \( x_k \to x_0 \) in \( X \), and put \( y_k = (x_0, x_0, \ldots, x_0, x_{k+1}, x_{k+2}, \ldots) \), where \( x_0 \) is repeated \( k \)-times,
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$k \in \mathbb{N}$. Then $\{y_k\}$ is a Cauchy sequence in $E_D$ and $y_k \to (x_0, x_0, \ldots)$ in the topology of $X^N$. Since $(x_0, x_0, \ldots) \notin E_D$, $D$ is not a Banach disk.

REFERENCES

3. QIU, Jing-Hui, Some Results on Bounded Sets in Inductive Limits, to appear.
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