ON NEW CLASSES OF ANALYTIC FUNCTIONS
WITH NEGATIVE COEFFICIENTS

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ABSTRACT. We introduce the classes $K^*_n$ of analytic functions with
negative coefficients by using the nth order Ruscheweyh derivative.
The object of the present paper is to show coefficient inequalities
and some closure theorems for functions $f(z)$ in $K^*_n$. Further we consider
the modified Hadamard product of functions $f_i(z)$ in $K^*_n$ ($n = 1, 2, \ldots, m$).

KEY WORDS AND PHRASES. Ruscheweyh derivative, Analytic functions, Negative
coefficients, coefficient inequalities and Hadamard product.

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I. INTRODUCTION.

Let $\mathcal{A}$ denote the class of functions $f(z)$ analytic in the unit disk
$U = \{z: |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. Ruscheweyh
[10] introduced the classes $K^*_n$ of functions $f(z) \in \mathcal{A}$ satisfying

$$
\Re \left\{ \frac{(z^n f(z))^{(n+1)}}{(z^{n-1} f(z))^{(n)}} \right\} > \frac{n + 1}{2} \tag{1.1}
$$

for $n \in \mathbb{N} \cup \{0\}$ and $z \in U$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Ruscheweyh [10]
showed the basic property

$$
K_{n+1} \subset K_n \tag{1.2}
$$

for each $n \in \mathbb{N} \cup \{0\}$. Note that $K_0$ is the class $S^*(1/2)$ of starlike
functions of order 1/2.

Let

$$
D^nf(z) = z(z^{n-1} f(z))^{(n)}/n! \tag{1.3}
$$

for $n \in \mathbb{N} \cup \{0\}$. This symbol $D^nf(z)$ was named the nth order Ruscheweyh
derivative of $f(z)$ by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f \ast g(z)$, that is, if $f(z)$ and $g(z)$ are given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$  \hspace{1cm} (1.4)

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n,$$  \hspace{1cm} (1.5)

respectively, then

$$f \ast g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$  \hspace{1cm} (1.6)

Using Hadamard product, Ruscheweyh [10] observed that if

$$D^\alpha f(z) = \frac{z}{(1 - z)^{\alpha + 1}} \ast f(z), \hspace{1cm} (\alpha \geq -1)$$  \hspace{1cm} (1.7)

then (1.3) is equivalent to (1.7) when $\alpha = n \in \mathbb{N} \cup \{0\}$.

Thus it follows from (1.1) that the necessary and sufficient condition for $f(z) \in A$ to belong to $K_n$ is

$$\Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \frac{1}{2}, \hspace{1cm} (z \in U).$$  \hspace{1cm} (1.8)

Note that $K_{-1}$ is the class of functions $f(z) \in A$ satisfying

$$\Re \left\{ \frac{f(z)}{z} \right\} > \frac{1}{2}, \hspace{1cm} (z \in U).$$  \hspace{1cm} (1.9)

For further information about the Hadamard products, the reader is advised to consult Ruscheweyh [11].

Recently many classes defined by using the nth order Ruscheweyh derivative of $f(z)$ were studied by Al-Amiri [2], [3], Bulboaca [4], Goel and Sohi [5], [6], Owa [8], [9], and Singh and Singh [13].
In this paper we introduce the following classes by using the nth order Ruscheweyh derivative of \( f(z) \). The method of proofs in section 2 follow closely the one used by Silverman [12]. Also several particular results obtained by Silverman [12] and Merkes, Robertson and Scott [7] can be deduced as special cases of our results in section 2.

**Definition.** We say that \( f(z) \) is in the class \( K^*_n (n \in \mathbb{N} \cup \{0\}) \), if \( f(z) \) defined by

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \quad (1.10)
\]

satisfies (1.8) for \( n \in \mathbb{N} \cup \{0\} \).

2. **COEFFICIENT INEQUALITIES AND APPLICATIONS.**

**Theorem 1.** Let the function \( f(z) \) be defined by (1.10). Then \( f(z) \) is in the class \( K^*_n \) if and only if

\[
\sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} a_k \leq (n + 1)! . \quad (2.1)
\]

Equality holds for the function defined by

\[
f(z) = z - \frac{(n + 1)!(k - 1)!}{(k + n - 1)!(2k + n - 1)} z^k, \quad (k \geq 2). \quad (2.2)
\]

**Proof.** We use a method of Silverman [12]. Assume that the inequality (2.1) holds. Then we have

\[
\left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| = \left| \frac{\sum_{k=2}^{\infty} (k + n - 1)(k + n - 2)\cdots k(k - 1)a_k z^{k-1}}{(n + 1)! - (n + 1) \sum_{k=2}^{\infty} (k + n - 1)(k + n - 2)\cdots k a_k z^{k-1}} \right|
\]
\[
\leq \frac{\sum_{k=2}^{\infty} (k + n - 1)(k + n - 2) \cdots k(k - 1)a_k |z|^{k-1}}{(n + 1)! - (n + 1) \sum_{k=2}^{\infty} (k + n - 1)(k + n - 2) \cdots ka_k |z|^{k-1}}
\leq \frac{\sum_{k=2}^{\infty} (k + n - 1)(k + n - 2) \cdots k(k - 1)a_k}{(n + 1)! - (n + 1) \sum_{k=2}^{\infty} (k + n - 1)(k + n - 2) \cdots ka_k}
\leq \frac{1}{2}.
\]

(2.3)

This shows that the values of \( D^{n+1}f(z)/D^n f(z) \) lie in a circle centered at \( w = 1 \) whose radius is 1/2. Consequently we can see that the function \( f(z) \) satisfies (1.8), hence further, \( f(z) \in K_n^* \).

For the converse, assume that the function \( f(z) \) is in the class \( K_n^* \). Then we get

\[
Re \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\}
= \left\{ \frac{(n + 1)! - \sum_{k=2}^{\infty} (k + n)(k + n - 1) \cdots ka_k z^{k-1}}{(n + 1)! - (n + 1) \sum_{k=2}^{\infty} (k + n - 1)(k + n - 2) \cdots ka_k z^{k-1}} \right\}
\geq \frac{1}{2}
\]

(2.4)

for \( z \in \mathbb{U} \). Choose values of \( z \) on the real axis so that \( D^{n+1}f(z)/D^n f(z) \) is real. Upon clearing the denominator in (2.4) and letting \( z \to 1^- \) through real values, we obtain

\[
(n + 1)! - \sum_{k=2}^{\infty} (k + n)(k + n - 1) \cdots ka_k
\geq \frac{1}{2} \left\{ (n + 1)! - (n + 1) \sum_{k=2}^{\infty} (k + n - 1)(k + n - 2) \cdots ka_k \right\}
\]

(2.5)

which implies (2.1).
Finally we can see that the function $f(z)$ defined by (2.2) is an extreme one for the theorem. This completes the proof of the theorem.

**Corollary 1.** Let the function $f(z)$ defined by (1.10) be in the class $\mathcal{K}_n^{*}$. Then

$$a_k \leq \frac{(n + 1)!(k - 1)!}{(k + n - 1)!(2k + n - 1)}$$

(2.6)

for $k \geq 2$. The equality holds for the function $f(z)$ of the form

$$f(z) = z - \frac{(n + 1)!(k - 1)!}{(k + n - 1)!(2k + n - 1)} z^k.$$  

(2.7)

**Theorem 2.** Let the function $f(z)$ defined by (1.10) be in the class $\mathcal{K}_n^{*}$. Then

$$|f(z)| \geq |z| - \left(\frac{1}{n + 3}\right)|z|^2$$

(2.8)

and

$$|f(z)| \leq |z| + \left(\frac{1}{n + 3}\right)|z|^2$$

(2.9)

for $z \in U$. The results are sharp.

**Proof.** Since $(k + n - 1)!(2k + n - 1)/(k - 1)!$ is increasing in $k$ ($k \geq 2$) and $f(z)$ is in the class $\mathcal{K}_n^{*}$, in view of Theorem 1, we obtain

$$(n + 1)!(n + 3) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} a_k$$

$$\leq (n + 1)!$$

(2.10)

which gives that

$$\sum_{k=2}^{\infty} a_k \leq \frac{1}{n + 3}.$$ 

(2.11)
Hence we can show that

\[ |f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \]

\[ \geq |z| - \left( \frac{1}{n + 3} \right)|z|^2 \quad (2.12) \]

and

\[ |f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \]

\[ \leq |z| + \left( \frac{1}{n + 3} \right)|z|^2 \quad (2.13) \]

for \( z \in \mathbb{U} \).

Further, by taking the function

\[ f(z) = z - \left( \frac{1}{n + 3} \right)z^2 \quad (2.14) \]

we can see that the results of the theorem are sharp.

**Corollary 2.** Let the function \( f(z) \) defined by (1.10) be in the class \( K_n^* \). Then \( f(z) \) is included in a disk with its center at the origin and radius \( r \) given by

\[ r = \frac{n + 4}{n + 3} \quad (2.15) \]

**Theorem 3.** Let the function \( f(z) \) defined by (1.10) be in the class \( K_n^* \). Then

\[ |f'(z)| \geq 1 - \left( \frac{2}{n + 3} \right)|z| \quad (2.16) \]

and

\[ |f'(z)| \leq 1 + \left( \frac{2}{n + 3} \right)|z| \quad (2.17) \]

for \( z \in \mathbb{U} \). The results are sharp.
PROOF. Note that \((k + n - 1)!/(k - 1)!\) is equal to \((k + n - 1)!k/(k - 1)!\) and \((k + n - 1)!/(k - 1)!\) is an increasing function of \(k\) \((k \geq 2)\). Hence, by virtue of Theorem 1, we have

\[
\frac{(n + 1)!}{2} \sum_{k=2}^{\infty} \frac{k a_k}{k} \leq \frac{(k + n - 1)!}{(k - 1)!} \sum_{k=2}^{\infty} \frac{a_k}{k}
\]

\[
\leq (n + 1)!
\]

which gives that

\[
\sum_{k=2}^{\infty} \frac{k a_k}{k} \leq \frac{2}{n + 3}
\]

Consequently, with the aid of (2.19), we can see that

\[
|f'(z)| \geq 1 - |z| \sum_{k=2}^{\infty} \frac{k a_k}{k}
\]

\[
\geq 1 - \left(\frac{2}{n + 3}\right) |z|
\]

and

\[
|f'(z)| \leq 1 + |z| \sum_{k=2}^{\infty} \frac{k a_k}{k}
\]

\[
\leq 1 + \left(\frac{2}{n + 3}\right) |z|
\]

for \(z \in \Omega\).

Further the bounds of the theorem are attained by the function \(f(z)\) given by (2.14).

**Corollary 3.** Let the function \(f(z)\) defined by (1.10) be in the class \(K_n^*\). Then \(f'(z)\) is included in a disk with its center at the origin and radius \(R\) given by

\[
R = \frac{n + 5}{n + 3}
\]

(2.22)
3. **CLOSURE THEOREMS.**

**Theorem 4.** Let the functions

\[ f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{k,i} \geq 0) \quad (3.1) \]

be in the class \( K_n^* \) for every \( i = 1, 2, 3, \ldots, m \). Then the function \( h(z) \) defined by

\[ h(z) = \sum_{i=1}^{m} c_i f_i(z) \quad (c_i \geq 0) \quad (3.2) \]

is also in the same class \( K_n^* \), where

\[ \sum_{i=1}^{m} c_i = 1. \quad (3.3) \]

**Proof.** By means of the definition of \( h(z) \), we can see that

\[ h(z) = z - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{m} c_i a_{k,i} \right) z^k. \quad (3.4) \]

Further, since \( f_i(z) \) are in \( K_n^* \) for every \( i = 1, 2, 3, \ldots, m \), we obtain

\[ \sum_{k=2}^{\infty} \frac{(k + n - 1)! (2k + n - 1)}{(k - 1)!} a_{k,i} \leq (n + 1)! \quad (3.5) \]

for every \( i = 1, 2, 3, \ldots, m \). Consequently we have

\[ \sum_{k=2}^{\infty} \frac{(k + n - 1)! (2k + n - 1)}{(k - 1)!} \left( \sum_{i=1}^{m} c_i a_{k,i} \right) \]

\[ = \sum_{i=1}^{m} c_i \left\{ \sum_{k=2}^{\infty} \frac{(k + n - 1)! (2k + n - 1)}{(k - 1)!} a_{k,i} \right\} \]

\[ \leq \left( \sum_{i=1}^{m} c_i \right) (n + 1)! \]

\[ = (n + 1)! \quad (3.6) \]

by using (3.5). This shows that the function \( h(z) \) belongs to the class \( K_n^* \). Thus we have the theorem.
Theorem 5. Let
\[ f_1(z) = z \] (3.7)
and
\[ f_k(z) = z - \frac{(k - 1)! (n + 1)!}{(k + n - 1)! (2k + n - 1)} z^k \] (3.8)
for \( k \in \mathbb{N} - \{1\} \). Then \( f(z) \) is in the class \( K^*_n \) if and only if it can be expressed in the form
\[ f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) , \] (3.9)
where \( \lambda_k \geq 0 \) for \( k \in \mathbb{N} \) and
\[ \sum_{k=1}^{\infty} \lambda_k = 1 . \] (3.10)

Proof. Suppose that
\[ f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \]
\[ = z - \sum_{k=2}^{\infty} \frac{(k - 1)! (n + 1)!}{(k + n - 1)! (2k + n - 1)} \lambda_k z^k . \] (3.11)

Then we get
\[ \sum_{k=2}^{\infty} \frac{(k + n - 1)! (2k + n - 1)}{(k - 1)!} \lambda_k \]
\[ = (n + 1)! \sum_{k=2}^{\infty} \lambda_k \]
\[ = (n + 1)! (1 - \lambda_1) \]
\[ \leq (n + 1)! . \] (3.12)
Thus we can see that $f(z)$ is in the class $K_n^*$ with the aid of Theorem 1.

Conversely, suppose that $f(z)$ is in the class $K_n^*$. Again, by (2.6), we obtain that

$$a_k \leq \frac{(k-1)!(n+1)!}{(k+n-1)!(2k+n-1)} \quad (3.13)$$

for $k \in \mathbb{N} - \{1\}$. Now, setting

$$\lambda_k = \frac{(k+n-1)!(2k+n-1)}{(k-1)!(n+1)!} a_k \quad (3.14)$$

for $k \in \mathbb{N} - \{1\}$ and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k \quad (3.15)$$

we have the representation (3.9). This completes the proof of the theorem.

4. **MODIFIED HADAMARD PRODUCT.**

Let $f(z)$ be defined by (1.10) and $g(z)$ be defined by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad (b_k \geq 0). \quad (4.1)$$

Further let $f*g(z)$ denote the modified Hadamard product of $f(z)$ and $g(z)$, that is,

$$f*g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. \quad (4.2)$$

**THEOREM 6.** Let the functions $f_i(z)$ defined by (3.1) be in the classes $K_n^*$ for each $i = 1, 2, 3, \ldots, m$, respectively. Then the modified Hadamard product $f_1*f_2*\ldots*f_m(z)$ belongs to the class $K_n^*$, where $n = \min \{n_1, \ldots, n_m\}$.
PROOF. We may suppose that \( n_1 = \text{Min} \{ n_i \} \). Then, by using \( f_i(z) \in K_{n_1}^*(i = 1, 2, 3, \ldots, m) \), we can know that (2.11) would imply
\[
a_{k,i} \leq \frac{1}{n_1 + 3} \quad (i = 2, 3, 4, \ldots, m) \quad (4.3)
\]
and
\[
\sum_{k=2}^{\infty} \frac{(k + n_1 - 1)!(2k + n_1 - 1)}{(k - 1)!} a_{k,1} \leq (n_1 + 1)! . \quad (4.4)
\]

Consequently, putting \( n_1 = n \) we can see that
\[
\sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} \left( \prod_{i=1}^{\infty} a_{k,i} \right)
\leq \prod_{i=2}^{m} \left( \frac{1}{n_1 + 3} \right) \sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} a_{k,1}
\leq (n + 1)! \prod_{i=2}^{m} \left( \frac{1}{n_1 + 3} \right)
\leq (n + 1)! . \quad (4.5)
\]

Hence we have the theorem.

COROLLARY 4. Let the functions \( f_i(z) \) defined by (3.1) be in the same class \( K_n^* \) for every \( i = 1, 2, 3, \ldots, m \). Then the modified Hadamard product \( f_1*f_2*\ldots*f_m(z) \) also belongs to the class \( K_n^* \).

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