RESEARCH NOTES

PERIODIC RINGS WITH COMMUTING NILPOTENTS

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ABSTRACT. Let $R$ be a ring (not necessarily with identity) and let $N$ denote the set of nilpotent elements of $R$. Suppose that (i) $N$ is commutative, (ii) for every $x$ in $R$, there exists a positive integer $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1}f(x)$, (iii) the set $I_n = \{x \mid x^n = x\}$ where $n$ is a fixed integer, $n > 1$, is an ideal in $R$. Then $R$ is a subdirect sum of finite fields of at most $n$ elements and a nil commutative ring. This theorem generalizes the $x^n = x$ theorem of Jacobson, and (taking $n = 2$) also yields the well-known structure of a Boolean ring. An Example is given which shows that this theorem need not be true if we merely assume that $I_n$ is a subring of $R$.

KEY WORDS AND PHRASES. Boolean ring, subdirect sum, subdirectly irreducible.

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1. INTRODUCTION.

A well known theorem of Jacobson [1] states that a ring $R$ satisfying the identity $x^n = x$, $n > 1$ is fixed, is a subdirect sum of finite fields of at most $n$ elements. Such rings, of course, have no nonzero nilpotents. With this as motivation, we consider the structure of a "periodic" ring $R$ with commuting nilpotents and for which the set $I_n = \{x \mid x^n = x\}$ forms an ideal in $R$. We show that such a ring $R$ has a structure similar to that given in Jacobson's Theorem. As a corollary, we show that by taking $n = 2$, we recover the familiar structure of a Boolean ring (as a subdirect sum of copies of $GF(2)$). Finally, we give an example which shows that this theorem need not be true if we assume that $I_n$ is merely a subring of $R$ (instead of an ideal).
2. MAIN RESULTS.

Our main result is the following

MAIN THEOREM. Let \( R \) be a ring (not necessarily with 1), and let \( N \) be the set of nilpotents of \( R \). Suppose that (i) \( N \) is commutative, (ii) for every \( x \) in \( R \), there exists a positive integer \( k = k(x) \) and a polynomial \( f(\lambda) = f_x(\lambda) \) with integer coefficients such that \( x^k = x^{k+1}f(x) \), (iii) the set
\[
I_n = \{ x \mid x \in R, \ x^n = x \} \quad \text{where} \quad n > 1, \quad \text{is an ideal in} \quad R.
\]
Then \( R \) is commutative and, in fact, \( R \) is a subdirect sum of fields of at most \( n \) elements and a nil commutative ring.

PROOF. The proof will be broken into several claims.

CLAIM 1. The idempotents of \( R \) are all in the center of \( R \).

For, suppose \( e^2 = e \in R \), \( x \in R \). Then \( e \in I_n \) and hence, by (iii),
\[
ex - exe \in I_n; \quad \text{that is},
\]
\[
ex - exe = (ex - exe)^n = 0
\]
and hence \( ex = exe \). Similarly, \( xe = exe \), which proves Claim 1.

CLAIM 2. If \( \phi: R \to R^* \) is an onto homomorphism, then \( \phi(N) \) coincides with the set of all nilpotent elements of \( R^* \).

This was proved by Abu-Khuzam and Yaqub [2] and by Ikehata and Tominaga [3], but for convenience we reproduce the proof. Let \( d^* \) be an arbitrary nilpotent element of \( R^* \) with \( (d^*)^m = 0 \). Let \( d \in R \) be such that \( \phi(d) = d^* \). By (ii), \( d^k = d^{k+1}f(d) \) for some positive integer \( k \) (depending on \( d \)) and some polynomial \( f(\lambda) \) with integer coefficients (again depending on \( d \)). The last equation implies that
\[
[d - d^2f(d)]^k = 0 \quad \text{and hence} \quad d - d^2f(d) \in N. \tag{2.1}
\]
Observe that
\[
d - d^{m+1}(f(d))^m = (d - d^2f(d)) + (df(d))(d - d^2f(d)) + ... + (df(d))^{m-1}(d - d^2f(d))
\]
and hence, by (2.1),
\[
d - d^{m+1}(f(d))^m \in N. \tag{2.2}
\]
Recalling that \( \phi(d) = d^* \) and \( (d^*)^m = 0 \), (2.2) implies
\[
d^* = \phi(d - d^{m+1}(f(d))^m) \in \phi(N); \quad \text{that is}, \quad d^* \in \phi(N). \quad \text{This proves Claim 2.}
\]

CLAIM 3. Hypothesis (ii) implies that \( (xf(x))^k \) is idempotent and, moreover,
\[
x^k = x^{k+1}f(x)^k.
\]
For, \( x^k = x^{k+1}f(x) \) implies (by multiplying both sides by \( xf(x) \) a suitable number of times) \( x^k = x^{k+r}(f(x))^r \) for all positive integers \( r \), and hence, in particular, \( x^k = x^{2k}(f(x))^k \). Note that \( (xf(x))^k \) is indempotent, and Claim 3 is proved.

To complete the proof of the Main Theorem, first recall that
\[
R \cong \text{a subdirect sum of rings } R_i(i \in \Omega);
\]
each \( R_i \) is subdirectly irreducible. Let
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\[ \phi_i : R \rightarrow R_i \]

be the natural homomorphism of \( R \) onto \( R_i \). We now distinguish two cases.

CASE 1: \( R_i \) does not have an identity. Let \( x_i \in R_i \) and let \( \phi(x) = x_i, x \in R \).

By Claims 3 and 1, \((xf(x))^k\) is a central idempotent in \( R \) and hence \((x_i f(x_i))^k\) is a central idempotent in the subdirectly irreducible ring \( R_i \): Therefore, \((x_i f(x_i))^k = 0\) and hence by Claim 3, \( x_i^k = 0 \). Thus, \( R_i \) is a nil ring. Moreover, by Claim 2,

\[ \phi_i(N) = \text{nilpotents of } R_i = R_i (\text{since } R_i \text{ is nil}). \quad (2.3) \]

But, by (i), \( N \) is commutative and hence by (2.3), \( \phi_i(N)[= R_i] \) is commutative. In other words, \( R_i \) is a nil commutative ring in this case.

CASE 2: \( R_i \) has an identity 1.

As we saw in Case 1, for any \( x_i \) in \( R_i \), \((x_i f(x_i))^k\) is a central idempotent and hence (since \( R_i \) is subdirectly irreducible)

\[ (x_i f(x_i))^k = 0 \text{ or } (x_i f(x_i))^k = 1. \quad (2.4) \]

If for some \( x_i \) in \( R_i \), \((x_i f(x_i))^k = 0\) then by Claim 3, \( x_i^k = 0 \) and thus \( x_i \) is nilpotent. On the other hand, if \((x_i f(x_i))^k = 1\) then \( x_i \) is a unit. We have thus shown that

\[ x_i \text{ is nilpotent or } x_i \text{ is a unit, for all } x_i, i \in R_i. \quad (2.5) \]

Let \( I_i^* = \phi_i(I_n) \). Then \( I_i^* \) is an ideal in \( R_i \). Let \( x_i \in I_i^* \) and thus \( x_i = \phi_i(x) \) for some \( x \in R \) with \( x^n = x \). Therefore,

\[ x_i^n = x_i \text{ for all } x_i \in I_i^*. \quad (2.6) \]

Let \( e \in R \) be such that \( \phi_i(e) = 1 \). By (ii), \( e^k = e^{k+1} f(e) \) and hence \( \phi_i(e^k) = \phi_i(e^{k+1} f(e)) \). Thus, \( 1 = 1 \cdot \phi_i(f(e)) = f(\phi_i(e)) = f(1) \). Moreover, since \((ef(e))^k\) is idempotent (Claim 3), \((ef(e))^k \in I_n \) and hence

\[ (\phi_i(e) \phi_i(f(e)))^k \in I_i^* = \phi_i(I_n). \quad (2.7) \]

Now, since \( \phi_i(e) = 1 \) and \( \phi_i(f(e)) = f(\phi_i(e)) = f(1) = 1 \) (as shown above), (2.7) implies that \( 1 \in I_i^* \) and hence \( R_i = I_i^* \) (since \( I_i^* \) is an ideal). Combining this with (2.6), we see that

\[ x_i^n = x_i \text{ for all } x_i \in R_i. \quad (2.8) \]

Combining (2.5) and (2.8), we conclude that \( R_i \) is a division ring satisfying the identity in (2.8), and hence by Jacobson's Theorem [1], \( R_i \) is a field with at most \( n \) elements (since \( n \) is fixed). This completes the proof of the Main Theorem.

Taking \( n = 2 \) in our Main Theorem, we get

COROLLARY 1. Let \( R \) be a ring and \( N \) the set of nilpotents of \( R \). Suppose that (i) \( N \) is commutative, (ii) for every \( x \) in \( R \), there exists a positive integer \( k \) and a polynomial \( f(\lambda) \) with integer coefficients such that \( x^k = x^{k+1} f(x) \), (iii) the idempotents of \( R \) form an ideal in \( R \). Then \( R \) is com-
mutative and, in fact, \( R \) is a subdirect sum of copies of \( \text{GF}(2) \) and a nil commutative ring.

As a further corollary of our Main Theorem, we obtain Jacobson's Theorem [1]:

**COROLLARY 2.** Let \( R \) be a ring satisfying the identity \( x^n = x \), where \( n > 1 \) is a fixed integer. Then \( R \) is a subdirect sum of finite fields each of which has at most \( n \) elements.

Taking \( n = 2 \) in Corollary 2, we also obtain the following

**COROLLARY 3.** A Boolean ring is a subdirect sum of copies of \( \text{GF}(2) \).

We conclude with the following example which shows that our Main Theorem need not be true if we merely assume that \( I_n \) is a subring of \( R \).

**EXAMPLE.** Let

\[
R = \begin{pmatrix}
a & b & c \\
0 & a^2 & 0 \\
0 & 0 & a
\end{pmatrix}
\end{pmatrix}, \quad a, b, c \in \text{GF}(4)
\]

Note that the set \( E \) of all idempotents is just \( \{0, 1\} \) and thus \( E \) is a subring of \( R \) (since \( R \) is of characteristic 2). It is readily verified that \( N^2 = \{0\} \) and \( x^2 = x^2 \) for all \( x \) in \( R \), and hence all the hypotheses of our Main Theorem are satisfied except that the subring \( E \) is not an ideal. Observe that \( R \) is not commutative.

**REFERENCES**


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