ABSTRACT. Let F be a non-trivial complete non-Archimedean valued field. Some locally F-convex topologies, on the space \( C_b(X,E) \) of all bounded continuous functions from a zero-dimensional topological space X to a non-Archimedean locally F-convex space E, are studied. The corresponding dual spaces are also investigated.

KEY WORDS AND PHRASES: non-Archimedean spaces, spherically complete, zero-dimensional, Banaschewski compactification, strict topologies.

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1. INTRODUCTION.

Several authors have studied various topologies on spaces of continuous functions with values into either a valued field or a non-Archimedean locally convex space. Some of the papers on the subject are the [1]-[9]. The strict topology was introduced for the first time by Buck [10] in the space \( C_b(X,E) \) of all bounded continuous functions from a locally compact space X to a locally convex space E. In recent years several other authors have extended Buck's results by generalizing the space X and taking E to be either the scalar field or a locally convex space or an arbitrary topological vector space. In [9] Prolla defined the strict topology in \( C_b(X,E) \) assuming that X is locally compact Hausdorff zero-dimensional and E a non-Archimedean normed space over a locally compact non-Archimedean valued field F. In [7] the author studied the strict topology \( \beta_0 \) on \( C_b(X,E) \) assuming that X is an arbitrary topological space and E a non-Archimedean locally F-convex space over a non-Archimedean valued field F.

In this paper we introduce and study the locally F-convex topologies \( \beta, \beta', \beta_1 \) and \( \beta'_1 \) on \( C_b(X,E) \) where X is zero-dimensional and E a non-Archimedean locally F-convex space. These topologies are defined by means of the Banaschewski compactification \( \beta_0 X \) of X and yield as corresponding dual spaces certain spaces of \( E' \)-valued measures.

2. PRELIMINARIES.

Throughout this paper, X will denote a Hausdorff zero-dimensiona1 (= ultraregular) topological space and \( \beta_0 X \) its Banaschewski compactification (see [1]). For a continuous
function $f$ from $X$ to an ultraregular topological space $Y$ for which $f(X)$ is relatively compact in $Y$, we will denote by $\hat{f}$ the unique continuous extension of $f$ to all of $\beta_0 X$. For various notions on non-Archimedean spaces we will refer to [11]-[13].

Let $F$ be a non-trivial complete non-Archimedean valued field and let $E$ be a Hausdorff non-Archimedean locally $F$-convex space over $F$. Let $C_b(X,E)$ denote the space of all bounded continuous $E$-valued functions on $X$ and let $C_c(X,E)$ be the subspace of those $f$ for which $f(X)$ is relatively compact in $E$. For a subset $A$ of $X$, we will denote by $\mathcal{X}_A$ the $F$-characteristic function of $A$. Also, if $f$ is a function from $X$ to $E$ and $p$ a seminorm on $E$, we will define $\|f\|_{A,p}$ and $\|f\|_p$ by

$$\|f\|_{A,p} = \sup \{p(f(x)) : x \in A\}, \quad \|f\|_p = \|f\|_{X,p}.$$ 

For an $F$-valued function $g$ on $X$, we define

$$\|g\|_A = \sup \{|g(x)| : x \in A\}, \quad \|g\|_X = \|g\|_{X}.$$ 

Let $\Gamma$ be an upwards directed family of continuous non-Archimedean seminorms on $E$ generating its topology. The uniform topology $\tau_u$ on a subspace of $C_b(X,E)$ is the locally $F$-convex topology generated by the family of non-Archimedean seminorms $f \mapsto \|f\|_p$, $p \in \Gamma$. The topology $\beta_0$, which was defined in [7], is the locally $F$-convex topology generated by the seminorms $p_q(f) = \|qf\|_p$ where $p \in \Gamma$ and $q$ is a bounded function from $X$ to $F$ which vanishes at infinity.

Let $S(X)$ be the algebra of all clopen subsets of $X$. We will denote by $M(X,E')$ (see [6]) the space of all finitely-additive $E'$-valued measures $m$ on $S(X)$ for which the set $m(S(X))$ is an equicontinuous subset of the dual space $E'$ of $E$. For each $m \in M(X,E')$ there exists $p \in \Gamma$ such that $m_p(x) \leq m(x)$ for all $x \in S(X)$.

As it is shown in [6], we have $m_p(A \cup B) = \max\{m_p(A), m_p(B)\}$. We will denote by $M(X,F)$ the space of all bounded finitely-additive $F$-valued measures on $S(X)$. If $m \in M(X,E')$, then, for each $s \in E$, the set function $m_s : S(X) \to E$, $(m_s(A) = m(A)s$, is in $M(X,F)$.

For a decreasing sequence $(A_n)$ (resp. net $(A_\alpha)$) of clopen subsets of $X$, we will write $A_n \downarrow \emptyset$ (resp. $A_\alpha \downarrow \emptyset$) if $\bigcap A_n = \emptyset$ (resp. $\bigcap A_\alpha = \emptyset$). An element $\mu$ of $M(X,F)$ is called $\sigma$-additive (resp. $\tau$-additive) if for each sequence $G_n \downarrow \emptyset$ (resp. net $G_\alpha \downarrow \emptyset$) of clopen subsets of $X$ we have $\lim \mu(G_n) = 0$ (resp. $\lim \mu(G_\alpha) = 0$). A member $m$ of $M(X,E')$ is called $\sigma$-additive (resp. $\tau$-additive) if each $m_s$, $s \in E$, is $\sigma$-additive (resp. $\tau$-additive). We will denote by $M_\sigma(X,E')$ and $M_\tau(X,E')$ the spaces of all $\sigma$-additive and all $\tau$-additive members of $M(X,E')$, respectively. For an $m \in M(X,F)$, we define $|m|$ on $S(X)$ by

$$|m|(A) = \sup \{|m(B)| : B \in S(X), B \subseteq \bar{A}\}.$$ 

Let now $m \in M(X,E')$ and $A \in S(X)$, $A \neq \emptyset$. Consider the family $\Omega_n$ of all $n = \{A_1, \ldots, A_n; x_1, \ldots, x_n\}$, where $A_1, \ldots, A_n$ is a clopen partition of $A$ and $x_j \in A_j$. The set $\Omega_n$ becomes directed by defining $\alpha_1 \geq \alpha_2$ if the partition of $A$ in $\alpha_1$ is a refinement of the partition in $\alpha_2$. If $f$ is an $E$-valued function on $X$ and $\alpha = (A_1, \ldots, A_n; x_1, \ldots, x_n) \in \Omega_n$, we define $\omega_\alpha(f, m) = \sum_{x_i} m(A_i)f(x_i)$. If the $\lim \omega_\alpha(f, m)$ exists, then we say that $f$ is $m$-integrable.
over $A$ and we denote this limit by $\int_A \operatorname{fdm}$. For $A = \emptyset$, we define $\int_A \operatorname{fdm} = 0$. We will write simply $\int_A \operatorname{fdm}$ for $\int_X \operatorname{fdm}$. It is shown in [6] that every $f \in \mathcal{C}_{cr}(X,E)$ is $m$-integrable over each $A \in \mathcal{S}(X)$. The function $T_m : \mathcal{C}_{cr}(X,E) \to F$, $T_m(f) = \int_A \operatorname{fdm}$, is linear and $\tau_u$-continuous.

Moreover, the mapping $T : \mathcal{M}(X,E') \to (\mathcal{C}_{cr}(X,E), \tau_u)'$, $T(m) = T_m'$, is linear one-to-one and onto. Hence, we may identify $\mathcal{M}(X,E')$ with the dual space of $(\mathcal{C}_{cr}(X,E), \tau_u)$.

Finally, we recall that a subset $A$ of a vector space over $F$ is called $F$-absolutely convex (or simply absolutely convex) if $\gamma A + \delta A \subseteq A$ for all $\gamma, \delta \in F$ with $|\gamma|, |\delta| \leq 1$.

3. THE STRICT TOPOLOGIES $\beta, \beta', \beta_1, \beta_1'$.

Before defining the topologies $\beta, \beta', \beta_1, \beta_1'$, we prove the following

**Lemma 3.1.** If $f_1, f_2$ are continuous $F$-valued functions on $X$, then there exists a continuous $F$-valued function $f$ in $X$ with $|f(x)| = \max(|f_1(x)|, |f_2(x)|)$ for all $x \in X$.

**Proof.** For each positive real number $r$, the set $\{s \in F : |s| = r\}$ is open in $F$. Hence, the set

$$A_1 = \{x \in X : |f_1(x)| = |f_2(x)| \neq 0\}$$

is open in $X$. Also open is the set

$$A_2 = \{x \in X : |f_1(x)| \neq |f_2(x)|\}.$$ 

Define $f$ on $X$ by

$$f(x) = f_1(x) \text{ if } x \in A_1 \quad f(x) = f_2(x) \text{ if } x \notin A_1.$$ 

It is easy to see that $|f(x)| = \max(|f_1(x)|, |f_2(x)|)$ for all $x \in X$. Also, $f$ is continuous. In fact $f$ is clearly continuous at each point of the open set $A_1 \cup A_2$. Suppose now that $f_1(x) = f_2(x) = 0$. Given $\epsilon > 0$, there exists a neighborhood $V$ of $x$ such that $|f_1(y)| < \epsilon$ for all $y \in V$, $i = 1, 2$. If $y \in V$, then $|f(y) - f(x)| = |f(y)| < \epsilon$ which proves that $f$ is continuous at $x$. Thus $f$ satisfies the requirements.

Let now $\Omega_1$ (respectively $\Omega$) denote the family of all $F$-zero (resp. compact) subsets of $\beta_0 X$ which are disjoint from $X$. For $A \in \Omega$, let $C_A$ denote the family of all $h \in \mathcal{C}_{cr}(X,F)$ such that $h|A = 0$. For each $p \in \Gamma$, let $\beta_{A,p}$ denote the locally $F$-convex topology on $C_A(X,E)$ generated by the family of non-Archimedean seminorms $\{p_\phi : \phi \in C_A\}$, where $p_\phi(f) = ||\phi f||_p$.

The locally $F$-convex topology $\beta_A$ is defined by the family of seminorms $\{p_\phi : p \in \Gamma, \phi \in C_A\}$. The topology $\beta_p$ (resp. $\beta_{1,p}$) is the locally $F$-convex inductive limit of the topologies $\beta_{A,p}$ $A \in \Omega$ (resp. $A \in \Omega_1$). The locally $F$-convex projective limit of the topologies $\beta_p$ (resp. $\beta_{1,p}$), $p \in \Gamma$, is denoted by $\beta'$ (resp. $\beta_1'$). Since $p \geq q$ implies $\beta_p \supseteq \beta_q$, we have $\beta' = \bigcup_{p \in \Gamma} \beta_p$. Analogously we have $\beta_1' = \bigcup_{p \in \Gamma} \beta_{1,p}$.

We define $\beta$ to be the locally $F$-convex inductive limit of the topologies $\beta_A$, $A \in \Omega$. Thus $\beta$ has as a base at zero the family of all $F$-absolutely convex subsets of $C_D(X,E)$ which are $\beta_A$-neighborhoods of zero for each $A \in \Omega$. Analogously, $\beta_1$ is the locally $F$-convex inductive limit of the topologies $\beta_A$, $A \in \Omega_1$.

We have the following easily established

**Lemma 3.2.** 1) The topologies $\beta, \beta', \beta_1$, and $\beta_1'$ are Hausdorff.
2) \( \beta' \leq \beta_1 \leq \gamma \) and \( \beta' \leq \beta \leq \beta_1 \).

**Lemma 3.3.** Let \( H \subseteq X \) and \( p \in \Gamma \). If \( (A_n) \) is a sequence of clopen subsets of \( X \) such that the closure \( \overline{A_n} \) in \( \beta_0 X \), of each \( A_n \), is disjoint from \( H \) and if \( 0 < a_n \to \infty \), then the set

\[
W_p(A_n, \alpha_n) = \bigcap_{n=1}^{\infty} \{ f \in C_b(X, E) : \| f \|_{A_n, p} \leq \alpha_n \}
\]

is a \( \beta_{H, p} \)-neighborhood of zero.

**Proof.** Assume that \( A_n \subseteq A_{n+1} \) for each \( n \). Set \( \gamma_n = \inf_{k \geq n} \alpha_k \). Then, \( 0 < \gamma_n \to \infty \) and \( \gamma_n \leq \gamma_{n+1} \). Clearly \( W_p(A_n, \gamma_n) \subseteq W_p(A_n, \alpha_n) \). Let \( \lambda \in \mathbb{F}, |\lambda| > 1 \). For each positive integer \( n \), there exists an integer \( m \) such that \( |\lambda|^m \leq \gamma_n < |\lambda|^{m+1} \). Take \( \lambda_n = \lambda^m \) (if \( m = 0 \), we take \( \lambda = 1 \)). Since \( |\lambda_n| > \gamma_n \cdot |\lambda|^{-1} \), we have \( |\lambda_n| \to \infty \). Also, \( |\lambda_n| \leq |\lambda_{n+1}| \) and \( |\lambda_n| \leq \gamma_n \). Moreover

\[
W_p(A_n, |\lambda_n|) \subseteq W_p(A_n, \gamma_n).
\]

Each \( \overline{A_n} \) is clopen in \( \beta_0 X \). Define \( h \) on \( \beta_0 X \) by

\[
h(x) = \begin{cases} 
1 & \text{if } x \notin \overline{A_1} \\
|\lambda_n|^{-1} & \text{if } x \in \overline{A_n} - \overline{A_{n-1}}, n \geq 2 \\
0 & \text{if } x \notin \bigcup_{n=1}^{\infty} \overline{A_n}.
\end{cases}
\]

Then \( h \) is continuous. In fact, \( h \) is clearly continuous on the open set \( B = \bigcup_{n=1}^{\infty} \overline{A_n} \). Let \( x_0 \in B \) and \( \varepsilon > 0 \). Choose \( n \) such that \( |\lambda_n| > 1/\varepsilon \). The set \( V_n = \beta_0 X - \overline{A_n} \) is a neighborhood of \( x_0 \). Moreover, for \( x \in V_n \), we have \( |h(x) - h(x_0)| = |h(x)| \leq |\lambda_n|^{-1} < \varepsilon \). So \( h \) is continuous at \( x_0 \). Also, \( h = 0 \) on \( H \). Set \( \psi = h |\lambda_n| \). We will show that

\[
W_p = \{ f \in C_b(X, E) : \| \psi f \|_{p, \alpha} \leq 1 \} \subseteq W_p(A_n, \alpha_n).
\]

In fact, let \( f \in W_p \) and \( x \in A_n \). If \( x \in \overline{A_1} \), then \( \psi(x) = 1 \) and so \( p(f(x)) \leq |\lambda_n| \leq |\lambda_n|^{-1} \). If \( x \notin \overline{A_1} \), then \( x \in \overline{A_k} - \overline{A_{k-1}} \) for some \( k \leq n \) and so \( \psi(x) = |\lambda_k|^{-1} \) which implies that \( p(f(x)) \leq |\lambda_k| \leq |\lambda_n|^{-1} \). Thus \( \| f \|_{A_n, p} \leq |\lambda_n|^{-1} \) for all \( n \) and this completes the proof.

**Theorem 3.4.** \( \beta_{H, p} \) has a base at zero the family of all sets of the form \( W_p(A_n, |\lambda|) \) where \( (A_n) \) is an increasing sequence of clopen subsets of \( X \), such that the closure \( \overline{A_n} \) in \( \beta_0 X \) is disjoint from \( H \), \( \lambda \in \mathbb{F} \) with \( |\lambda_n| \leq |\lambda_{n+1}| \) and \( 0 < |\lambda| \to \infty \).

**Proof.** Using Lemma 3.1, we get that \( \beta_{H, p} \) has as a base at zero the family of all sets of the form \( W_p, h = \{ f \in C_b(X, E) : \| h f \|_{p, \alpha} \leq 1 \} \) where \( h \in C \). Let now \( h \in C \) and \( W = W_p, h \). Let \( \lambda \in \mathbb{F} \) with \( |\lambda| > 1 \), \( ||h|| \). Set

\[
A_n = \{ x \in X : |h(x)| \geq |\lambda|^{-n} \}.
\]

Clearly \( A_n \) is clopen, \( A_n \subseteq A_{n+1} \) and \( A_n \subseteq \{ x \in \beta_0 X : |h(x)| \geq |\lambda|^{-n} \} \) and so \( A_n \) is disjoint from \( H \). Set \( W_1 = W_p(A_n, |\lambda|) \) where \( \lambda_1 = |\lambda|^{-1} \) and \( \lambda_n = \lambda_{n-1}^{-1} \) if \( n \geq 2 \). We will show that \( W_1 \subseteq W \). In fact, let \( f \in W_1 \). We need to show that

\[
(*) \quad p(h(x)f(x)) \leq 1
\]

for all \( x \in X \). Clearly \( (*) \) holds if \( x \notin \bigcup_{n=1}^{\infty} A_n \). If \( x \in A_n \), then \( p(f(x)) \leq |\lambda|^{-n} \) and so \( (*) \)
holds since $|h(x)| < |\lambda|$. Finally, if $x \in A_{m^{-1}}$, $m > 1$, then $|h(x)| < |\lambda|^{-m+1}$ and $p(h(x)f(x)) \leq |\lambda|^{-m}$ which implies that $p(h(x)f(x)) \leq 1$. This proves that $W_1 \subseteq W$. This and the preceding Lemma complete the proof.

**Theorem 3.5.** (i) $\beta_0 \leq \beta' \leq \beta$.

(ii) The topologies $\beta_0, \beta, \beta', \beta_1, \beta'_1$ and $\tau_u$ have the same bounded sets.

(iii) If $X$ is locally compact, then $\beta_0 = \beta'$.

**Proof.** Let $W$ be an absolutely convex $\beta_0$-neighborhood of zero. There exist $p \in \Gamma$ and a bounded $\Gamma$-valued function $h$ on $X$ vanishing at infinity such that $W_1 = \{ f \in C_b(X, \Gamma) : \|h f\|_p \leq 1 \} \subseteq W$. By [7, Proposition 2.6], there exists a sequence $(K_n)$ of compact subsets of $X$ and $0 < \alpha \to \infty$ such that $W(K_n, \alpha, \beta_0, \beta_1) \subseteq W$. Let $H \in \mathcal{H}$. For each $n$, there exists a clopen subset $B_n$ of $\beta_0 X$ containing $K_n$ and disjoint from $H$. If $A_n = B_n \cap X$, then $W(A_n, \alpha, \beta_0, \beta_1)$ is a $\beta_0, \beta_1$-neighborhood of zero contained in $W_1$. Thus $W$ is a $\beta_0, \beta_1$-neighborhood of zero for every $H \in \mathcal{H}$ and hence $W$ is a $\beta_0, \beta_1$-neighborhood of zero.

(ii) It follows from (i) and from Lemma 2.2, since $\beta_0$ and $\tau_u$ have the same bounded sets (see [7, Proposition 2.11]).

(iii) If $X$ is locally compact, then the set $H \cap \beta_0 X = \beta_0 X$ is closed in $\beta_0 X$ (see [14,XI, 8.3]). If $V$ is a $\beta_0$-neighborhood of zero, then $V$ is a $\beta_0$-neighborhood of zero and hence a $\beta_0$-neighborhood of zero since every member of $\mathcal{C}_H$ vanishes at infinity. Thus $\beta_0 \leq \beta$ and so $\beta = \beta_0$.

4. THE STRICT TOPOLOGIES ON $CO_c(X, \Gamma)$.

Throughout the rest of the paper, we will consider the strict topologies on the subspace $CO_c(X, \Gamma)$ of $C_b(X, \Gamma)$.

**Theorem 4.1.** Let $p \in \Gamma$, $H \in \mathcal{H}$ and $V$ an absolutely convex subset of $CO_c(X, \Gamma)$. Then $V$ is a $\beta_0, \beta_1$-neighborhood of zero iff the following condition is satisfied: For each $d > 0$ there exist a clopen subset $A$ of $X$ whose closure $\overline{A}$ in $\beta_0 X$ is disjoint from $H$, and $\delta > 0$ such that

$$\{ f \in CO_c(X, \Gamma) : \|f\|_p \leq d, \|f\|_{A_0, p} \leq \delta \} \subseteq V.$$  

**Proof.** Suppose that $V$ is a $\beta_0, \beta_1$-neighborhood of zero. By Theorem 3.4, there exist an increasing sequence $(\lambda_n)$ of clopen subsets of $X$, with $\overline{A_n} \cap H = \emptyset$, and $\lambda_n \in \Gamma$ with $0 < \lambda_n \downarrow \infty$, $|\lambda_n| \leq |\lambda_{n+1}|$, such that $W(p(A_n, \lambda_n)) \subseteq V$. Let now $d > 0$ and choose $n$ such that $|\lambda_k| > d$ if $k > n$. Take $\delta = |\lambda_n|$ and $A = A_n$. It is clear that if $f \in CO_c(X, \Gamma)$ is such that $\|f\|_p \leq d$ and $\|f\|_{A_0, p} \leq \delta$, then $f \in W(p(A_n, \lambda_n)) \subseteq V$.

Conversely, assume that the condition is satisfied. Let $\lambda \in \Gamma$, $|\lambda| > 1$. Choose an increasing sequence $(A_n)$ of clopen sets, with $\overline{A_n} \cap H = \emptyset$, and a decreasing sequence $(\delta_n)$ of positive numbers such that $U_n \cap \lambda^n \subseteq CV$, where

$$U_n = \{ f \in CO_c(X, \Gamma) : \|f\|_{A_0, p} \leq \delta_n \}.$$  

Set

$$V_1 = U \cap \left[ \bigcap_{n=1}^\infty (U_n + \lambda^n U) \right].$$
We will show that \( V_1 \subseteq V_1 \). In fact, let \( f \in V_1 \). Then \( f \in U_1 \) and, for each \( n \), \( f = g_n + h_n \) with \( g_n \in \lambda^N \), \( h_n \in U_{n+1} \). Let \( N \) be such that \( f \in \lambda^N \). Set \( f_1 = g_1 \) and \( f_k = g_k - g_{k-1} \) if \( k > 1 \). We have \( f = f_1 + f_2 + \ldots + f_N \). Since \( f_1 = g_1 \in U \) and \( f_k = h_k \in \lambda^N \), \( U_1 \subseteq U_1 \), \( U_1 \subseteq U_1 \), we have \( f_1 \in U_1 \cap \lambda^N \subseteq V \). For \( n > 1 \), we have \( f_n = g_n - g_{n-1} \in \lambda^N \) and \( h_n \in U_1 \subseteq U_1 \). Also, \( h_n = h_{n-1} \in U_1 \subseteq U_1 \) and hence \( f_n \in U_1 \cap \lambda^N \subseteq V \). Finally, \( h_n = f_n - g_n \in \lambda^N \subseteq U_1 \) and \( h_n \in U_1 \subseteq U_1 \), so again \( h_n \in V \). It follows that \( f \in V \) since \( V \) is absolutely convex. Thus, \( V_1 \subseteq V \). Let now \( \lambda_1 \in F \) with \( 0 < |\lambda_1| < 1 \), \( \delta_1 \) and \( \lambda_n = \lambda^{n+1} \) if \( n > 1 \). We will finish the proof by showing that

\[
V_2 = \bigcap_{n=1}^{\infty} \{ f \in C_{rc}(X,E) : \| f \|_{A_n,p} \leq |\lambda_n| \} \subseteq V_1.
\]

So, let \( f \in V_2 \). Since \( \| f \|_{A_n,p} \leq |\lambda_n|^{n+1} \), we have \( f \in U_1 \). Let \( m \) be any positive integer. Since \( f(A_{m+1}) \) is relatively compact, there are \( x_1, \ldots, x_n \in A_{m+1} \) such that

\[
f(A_{m+1}) \subseteq \bigcup_{i=1}^{n} \{ s : p(s-f(x_i)) \leq 1 \}
\]

and so

\[
A_{m+1} \subseteq \bigcup_{i=1}^{n} G_i = G
\]

where \( G_i = \{ x \in X : p(f(x) - f(x_i)) \leq 1 \} \). Clearly \( G \) is clopen. Moreover, if \( x \in G \), then \( p(f(x)) \leq \max_{i} (p(f(x_i))) \leq |\lambda|^m \). Let \( g \in \mathcal{C}_G \), \( h = f-g \). Then \( h = 0 \) on \( A_{m+1} \) and so \( h \in U_{m+1} \). Also, \( \| g \|_{p} \leq |\lambda|^m \) and so \( g \in \lambda^m \). This proves that \( f \in V_1 \) and so the result follows by Lemma 3.3.

For \( p \in \Gamma \), let \( \beta_p \) denote the locally \( F \)-convex topology generated by the non-Archimedean seminorms \( f \mapsto \| f \|_p \).

**Theorem 4.2.** (i) For \( H \in \mathcal{G} \), \( \beta_{H,p} \) is the finest locally \( F \)-convex topology on \( C_{rc}(X,E) \) which agrees with \( \beta_{H,p} \) on \( p \)-bounded sets.

(ii) \( \beta_p \) (resp. \( \beta_{1,p} \)) is the finest locally \( F \)-convex topology on \( C_{rc}(X,E) \) which agrees with \( \beta_p \) (resp. \( \beta_{1,p} \)) on \( p \)-bounded sets.

**Proof.** (i) Let \( \tau \) be a locally \( F \)-convex topology on \( C_{rc}(X,E) \) which agrees with \( \beta_{H,p} \) on \( p \)-bounded sets and let \( V \) be an absolutely convex \( \tau \)-neighborhood of zero. Given \( d > 0 \)

there exists a \( \beta_{H,p} \)-neighborhood \( V_1 \) of zero such that

\[
V_1 \cap \{ f \in C_{rc}(X,E) : \| f \|_p \leq d \} \subseteq V.
\]

By Theorem 4.1, there exist a clopen set \( A \) in \( X \), whose closure in \( \beta_{0}X \) is disjoint from \( H \), and \( \delta > 0 \) such that

\[
\{ f \in C_{rc}(X,E) : \| f \|_p \leq d, \| f \|_{A,p} \leq \delta \} \subseteq V_1.
\]

Thus

\[
\{ f \in C_{rc}(X,E) : \| f \|_p \leq d, \| f \|_{A,p} \leq \delta \} \subseteq V.
\]

This, by the preceding Theorem, implies that \( V \) is a \( \beta_{H,p} \)-neighborhood of zero. Thus

\[
\tau \leq \beta_{H,p}.
\]

(ii) It follows easily from (i).
5. DUAL SPACES FOR THE STRICT TOPOLOGIES.

Since each of the topologies $\beta, \beta', \beta_1, \beta_1'$ is coarser than $\tau_u$ and since (by [6])
$(C_{rc}(X,E), \tau_u)' = M(X,E')$, it follows that the dual space of $C_{rc}(X,E)$ under any one of the
topologies $\beta, \beta', \beta_1, \beta_1'$ is a subspace of $M(X,E')$.

**THEOREM 5.1.** (i) $(C_{rc}(X,E), \beta)' \subseteq M_\tau(X,E')$.
(ii) $(C_{rc}(X,E), \beta_1)' \subseteq M_\sigma(X,E')$.

**PROOF.** (i) Let $m \in M(X,E')$ be in the dual space of $(C_{rc}(X,E), \beta)$ and let $s \in E$. Given
$\epsilon > 0$, the set

$$W = \{ f \in C_{rc}(X,E) : \left\| \int f \, dm \right\| \leq \epsilon \}$$

is a $\beta$-neighborhood of zero. Let now $(A_\alpha)$ be a net of clopen subsets of $X$ with $A_\alpha \not\to \emptyset$.
The closure $B_\alpha$ of $A_\alpha$ in $B_0X$ is clopen and $B_\alpha + Q \subseteq E$. Since $W$ is a $B_0$-neighborhood of zero,
there exist $h \in C_Q$ and $p \in \Gamma$ such that

$$W = \{ f \in C_{rc}(X,E) : \left\| hf \right\|_p \leq 1 \} \subseteq W.$$  

Choose $\delta > 0$ such that $\delta \cdot p(s) \leq 1$ and set

$$B = \{ x \in B_0X : \left| h(x) \right| \leq \delta \}.$$  

Since $B_0X - B$ is compact, there exists $\delta > 0$ with $B_\alpha \subseteq B$. Let now $\alpha \geq \alpha_0$. If $f = \chi_{\alpha_\alpha} s$, then
$f \in W$ and so

$$\left| m(A_\alpha) s \right| = \left\| \int f \, dm \right\| \leq \epsilon.$$  

This proves that $\lim m(A_\alpha) s = 0$ for every $s \in E$ and so $m \in M_\tau(X,E')$.

(ii) The proof is analogous to that of (i).

**THEOREM 5.2.** $(C_{rc}(X,F), \beta_1)' = M_\sigma(X,E')$.

**PROOF.** By the preceding Theorem, it suffices to show that if $m \in M_{rc}(X,F)$, then the
mapping $f \mapsto \int f \, dm$ is $\beta_1$-continuous on $C_{rc}(X,F)$. So, let $m \in M_{rc}(X,F)$ and set

$$W = \left\{ f \in C_{rc}(X,F) : \left\| \int f \, dm \right\| \leq 1 \right\}.$$  

Let $Q \subseteq \Omega_1$. There exists a decreasing sequence $(B_n)$ of clopen sets in $\beta_0X$ with $Q = \cap B_n$.
Let $A_n = B_n \cap X$. Since $A_n + \emptyset$, we have $\left| m(A_n) \right| \to 0$ (see [6, Theorem 3.2]). Let now $d > 0$ and
choose $\lambda, \mu$ in $F$ with $\left| \lambda \right| \geq d$, $\left| \mu \right|$, $\left| m(X) \right| \leq 1$, $\mu \not\to 0$. Choose $n$ such that $\left| m(A_n) \right| < \left| \lambda \right|^{-1}$ and
take $A = X - A_n$. Clearly $A$ is clopen and its closure in $\beta_0X$ is contained in $\beta_0X - B_n$ and so
it is disjoint from $Q$. Let now $f \in C_{rc}(X,F)$ with $\left\| f \right\| \leq d$ and $\left\| f \right\|_A \leq \left| \mu \right|$. Then

$$\left\| \int f \, dm \right\| \leq \left| \lambda \right| \cdot \left| m(A_n) \right| \leq 1 \text{ and } \left\| \int f \, dm \right\| \leq \left| \mu \right| \cdot \left| m(A) \right| \leq 1.$$  

Hence

$$\left\| \int f \, dm \right\| \leq \max \left\{ \left\| \int f \, dm \right\|, \left\| \int f \, dm \right\|_A \right\} \leq 1$$

which proves that $f \in W$. By Theorem 4.1, $W$ is a $B_0$-neighborhood of zero. Since this is
true for all $Q \subseteq \Omega_1$ and since $W$ is absolutely convex, it follows that $W$ is a $B_1$-neighborhood
of zero and so \( m \in (C_{rc}(X,F), \beta_1) \).

**DEFINITION 5.3.** Let \( H \subseteq M(X,E) \). Then, \( H \) is called:

(i) uniformly \( \sigma \)-additive iff the following condition is satisfied: If \( (A_n) \) is a sequence of clopen sets with \( A_n + \emptyset \), then \( m(A_n) \rightarrow 0 \) uniformly for \( m \in H \).

(ii) uniformly \( \tau \)-additive iff the following condition is satisfied: If \( (A_n) \) is a net of clopen subsets of \( X \) with \( A_n + \emptyset \), then \( m(A_n) \rightarrow 0 \) uniformly for \( m \in H \).

**THEOREM 5.4.** Let \( H \subseteq M(X,F) \). Then:

(i) \( H \) is uniformly \( \tau \)-additive iff \( |m|(A_n) \rightarrow 0 \) uniformly for \( m \in H \) whenever \( A_n + \emptyset \).

(ii) \( H \) is uniformly \( \sigma \)-additive iff for each sequence \((A_n)\) of clopen subsets of \( X \) with \( A_n + \emptyset \), we have \( |m|(A_n) \rightarrow 0 \) uniformly for \( m \in H \).

**PROOF.** (i) The condition is clearly sufficient. Conversely, assume that \( H \) is uniformly \( \tau \)-additive and let \( A_\alpha + \emptyset \). Suppose, by way of contradiction, that there exists \( \epsilon > 0 \) such that \( \sup_{m \in H} |m|(A_\alpha) > \epsilon \) for all \( \alpha \). Let \( A_0 \) be fixed and choose \( m \in H \) with \( |m|(A_0) > \epsilon \). Since \( A_0 \subseteq (X-B_0) + \emptyset \), there exists \( A_1 \supseteq A_0 \) such that \( |m(A_1) \cap (X-B_0)| \leq \epsilon \). Let \( B_1 = B_0 \cup A_1 \). Then, \( A_1 \subseteq B_1 \subseteq A_\alpha \). Moreover, since \( |m(B_0)| > \epsilon \) and \( |m(A_1) \cap (X-B_0)| \leq \epsilon \) and since \( m(B_1) = m(B_0) + m(\emptyset) \subseteq B_1 \subseteq A_\alpha \), we have \( |m(B_1)| = |m(B_0)| > \epsilon \). Thus, for each \( \alpha \) there exist \( A_1 \supseteq A_\alpha \), \( m \in H \) and clopen set \( B \) with \( A_1 \subset B \subseteq A_\alpha \) and \( |m(B)| > \epsilon \). Let \( D \) denote the set of all \( B \in X \) with the following property: There are \( A_1, A_2, A_2 \supseteq A_\alpha \), and \( m \in H \) such that \( A_1 \subset B \subseteq A_\alpha \) and \( |m(B)| > \epsilon \). For each \( \alpha \) there exists \( B \in X \) with \( B \subseteq A_\alpha \) and \( |m(B)| > \epsilon \). For each \( \alpha \) there exists \( B \in X \) with \( B \subseteq A_\alpha \) and \( |m(B)| > \epsilon \). And \( D \) is directed downwards to the empty set. Since, for each \( B \in X \) there exists \( m \in H \) with \( |m(B)| > \epsilon \), it follows that \( H \) is not uniformly \( \tau \)-additive and this contradiction completes the proof of (i).

(ii) Suppose that there exists a sequence \((A_n)\) of clopen subsets of \( X \) and \( \epsilon > 0 \) such that \( \sup_{m \in H} |m|(A_n) > \epsilon \) for each \( n \). We will show that for each \( n \) there exist \( k > n, m \in H \) and \( A_k \subseteq B \subseteq A_n \) with \( |m(B)| > \epsilon \). In fact, there exist \( m \in H \) and \( B \subseteq A_n \) with \( |m(B)| > \epsilon \). Since \( A_k \cap (X-B_0) + \emptyset \), there exists \( k > n \) with \( |m(A_k \cap (X-B_0))| \leq \epsilon \). Now it suffices to take \( B = A_k \). We get now inductively a sequence of indices \( n_1 < n_2 < \ldots \), a sequence \((B_i)\) of clopen sets and a sequence \((m_i)\) in \( H \) such that \( A_n \subseteq B_i \subseteq A_n \) and \( |m_i(B_i)| > \epsilon \). Since \( B_i + \emptyset \), \( H \) is not uniformly \( \sigma \)-additive. It is clear now that the result of (ii) follows.

**COROLLARY 5.5.** Let \( m \in M(X,F) \). Then \( m \) is \( \tau \)-additive iff \( |m|(A) \rightarrow 0 \) whenever \( A + \emptyset \).

**THEOREM 5.6.** Let \( H \subseteq M(X,F) \). Then:

(i) \( H \) is an equicontinuous subset of the dual space of \( (C_{rc}(X,F), \beta) \) iff \( H \) is norm-bounded (i.e. \( \sup_{m \in H} |m|(X) < \infty \)) and uniformly \( \tau \)-additive.

(ii) \( H \) is an equicontinuous subset of the dual space of \( (C_{rc}(X,F), \beta_1) \) iff \( H \) is norm bounded and uniformly \( \sigma \)-additive.
PROOF. (i) Suppose that \( H \) \( \beta \)-equicontinuous. Then the polar \( H^0 \) of \( H \) in \( C_{\text{rc}}(X,F) \) is a \( \beta \)-neighborhood of zero and hence a \( \tau \)-neighborhood of zero. Thus there exists \( \lambda \neq 0 \) in \( F \) such that
\[
W = \{ f \in C_{\text{rc}}(X,F) : ||f|| \leq |\lambda| \} \subset H^0.
\]
If now \( A \in S(X) \), then \( \lambda \chi_A \in W \) and so \( |m(A)| \leq |\lambda|^{-1} \) for all \( m \in H \). It follows that \( |m|(x) \leq |\lambda|^{-1} \) for all \( m \in H \) and so \( H \) is norm-bounded. Let now \( A_\alpha \neq \emptyset \). If \( B_\alpha = \overline{A_\alpha} \) is the closure of \( A_\alpha \) in \( B_0X \), then \( B_\alpha + Q \in \Omega \). Since \( H^0 \) is a \( \beta \)-neighborhood of zero, there exists \( h \in C_Q \) such that
\[
W_1 = \{ f \in C_{\text{rc}}(X,F) : ||hf|| \leq 1 \} \subset H^0.
\]
Let \( \epsilon > 0 \) and choose \( \mu \neq 0 \) in \( F \) with \( |\mu| \leq \epsilon \). The set
\[
C = \{ x \in B_0X : |\widehat{h}(x)| \leq |\mu| \}
\]
is clopen and contains \( Q \). Since \( B_\alpha + Q \), there exists \( a_0 \) such that \( B_{a_0} \subset C \). If \( \alpha \geq a_0 \), then
\[
u^{-1} \chi_{A_\alpha} \in W_1 \text{ and so } |\nu|^{-1} |m(A_\alpha)| \leq 1 \text{ for all } m \in H \text{. Thus } |m(A_\alpha)| \leq |\mu| \leq \epsilon \text{ for all } m \in H \text{ and all } \alpha \geq a_0 \text{. This proves that } H \text{ is uniformly } \tau \text{-additive.}
\]
Conversely, suppose that \( H \) is norm-bounded and uniformly \( \tau \)-additive. Let \( d > 0 \). Choose \( \lambda \in F \) with \( |\lambda| \geq d \) and a non-zero \( \gamma \in F \) such that \( |\gamma| \cdot |m|(x) \leq 1 \) for all \( m \in H \). Let \( Q \in \Omega \). There exists a decreasing net \( (B_\alpha) \) of clopen sets in \( B_0X \) with \( \bigcap B_\alpha = Q \). If \( A_\alpha = B_\alpha \cap X \), then \( A_\alpha \neq \emptyset \). By hypothesis and by Theorem 5.4 there exists \( a \) such that \( |m|(A_\alpha) \leq |\lambda|^{-1} \) for all \( m \in H \). Let \( D = X - A_\alpha \). Then \( D \) is clopen and its closure in \( B_0X \) is disjoint from \( Q \).
If now \( f \in C_{\text{rc}}(X,F) \) is such that \( ||f|| \leq d \) and \( ||f||_D \leq |\mu| \), then, for all \( m \in H \), we have
\[
\int_{A_\alpha} f dm \leq |\lambda| \cdot |m|(A_\alpha) \leq 1, \quad \int_{D} f dm \leq |\mu| \cdot |m|(D) \leq 1
\]
and so \( ||f|| \leq 1 \). It follows that
\[
(f \in C_{\text{rc}}(X,F) : ||f|| \leq d, \quad ||f||_D \leq |\mu|) \subset H^0.
\]
By Theorem 4.1, \( H^0 \) is a \( \beta \) \( Q \)-neighborhood of zero for all \( Q \in \Omega \) and so \( H^0 \) is a \( \beta \)-neighborhood of zero which implies that \( H \) is \( \beta \)-equicontinuous.

(ii) The proof is analogous to that of (i).

Using the preceding Theorem and Theorem 5.1. we get the following

THEOREM 5.7. \( (C_{\text{rc}}(X,F), \beta') = M_\tau(X,F) \).

THEOREM 5.8. Let \( HCM(X,E') \) and \( p \in F \). The following are equivalent:

(i) \( H \) is an equicontinuous subset of the dual space of \( (C_{\text{rc}}(X,E), \beta_p) \).

(ii) a) \( \sup_{m \in H} m(X) < \infty \).

b) If \( A_\alpha \neq \emptyset \), then \( m(A_\alpha) \to 0 \) uniformly for \( m \in H \).

(iii) The set \( H = \{ m : m \in H, p(s) \geq 1 \} \) is norm bounded and uniformly \( \tau \)-additive.

(iv) \( H_p \) is an equicontinuous subset of the dual space of \( (C_{\text{rc}}(X,E), \beta) \).

PROOF. (i \( \Rightarrow \) ii) Since \( \beta \leq u \), \( H \) is \( u \)-equicontinuous and from this follows that
\[
\sup_{m \in H} m(X) < \infty \text{. Let now } A_\alpha \neq \emptyset \text{ and let } B_\alpha = \overline{A_\alpha} \text{ be the closure of } A_\alpha \text{ in } B_0X \text{. Then } B_\alpha + Q \in \Omega \text{.}
\]

(iii) If \( m \in H \), then \( m(A_\alpha) \leq 1 \).

(iv) \( H_p \) is an equicontinuous subset of the dual space of \( (C_{\text{rc}}(X,E), \beta) \).
There exists $h \in C_0$ such that 

$$\mathcal{W}_1 = \{ f \in C_{rc}(X, E) : \|h\|_p \leq 1 \} \subseteq H_0^1.$$

Let $\varepsilon > 0$ and choose $\lambda \neq 0$ in $F$ with $|\lambda| \leq \varepsilon$. There exists $a_0$ such that 

$$A_{a_0} \subseteq \{ x \in X : |h(x)| \leq |\lambda| \}.$$

Let now $a \geq a_0$ and $B \in S(X)$ with $B \subseteq A_a$. If $p(s) \leq 1$, then $\lambda^{-1} \cdot \chi_{B \cup A} \in \mathcal{W}_1$ and so $|m(A) s| \leq |\lambda| \leq \varepsilon$ for all $m \in \mathcal{H}$. It follows that $m_p(A) \leq \varepsilon$ for all $m \in \mathcal{H}$ and all $a \geq a_0$.

(ii $\Rightarrow$ i) It follows by an argument analogous to the one used in the proof of Theorem 5.8.

By Theorem 5.8, (iii) is equivalent to (iv). Finally, it is easy to see that (ii) is equivalent to (iii).

The proof of the following Theorem is analogous to the one of the preceding Theorem.

**THEOREM 5.9.** Let $H \subseteq M(X, E')$ and $p \in \mathcal{T}$. The following are equivalent:

(i) $H$ is an equicontinuous subset of the dual space of $(C_{rc}(X, E), \beta_1^\prime, p)$.

(ii) a) $\sup_{\mathcal{M}} m_p(A) < \infty$.

b) If $(A_n)$ is a sequence in $S(X)$ with $A_n \neq \emptyset$, then $m_p(A_n) \to 0$ uniformly for $m \in \mathcal{H}$.

(iii) The set $H_p = \{ m : m \in \mathcal{M}, p(m) \leq 1 \}$ is norm-bounded and uniformly $\sigma$-additive.

(iv) $H_p$ is an equicontinuous subset of the dual space of $(C_{rc}(X, F), \beta_1^\prime)$.

For $p \in \mathcal{T}$, let $M_p'(X, E')$ (resp. $M_\mathcal{M}_p'(X, E')$) be the set of those $m \in M(X, E') = \{ \mu \in \mathcal{M}(X, E') : \mu(X) < \infty \}$ for which for each sequence $(A_n)$ (resp. net $(A_a)$) of clopen sets in $X$ with $A_n \neq \emptyset$ (resp. $A_a \neq \emptyset$) we have $m(A_n) \to 0$ (resp. $m(A_a) \to 0$). Let 

$$M_\mathcal{M}'(X, E') = \bigcup_{p \in \mathcal{T}} M_p'(X, E'), M_\mathcal{M}'(X, E') = \bigcup_{p \in \mathcal{T}} M_\mathcal{M}_p'(X, E').$$

By Theorem 5.8 and 5.9, we have the following

**THEOREM 5.10.** $(C_{rc}(X, E), \beta^\prime, p)' = M_\mathcal{M}'(X, E')$ and $(C_{rc}(X, E), \beta_1^\prime, p)' = M_p'(X, E')$.

**THEOREM 5.11.** Suppose that $F$ is spherically complete and that $E$ is a non-Archimedean normed space over $F$. Then:

(i) If $(f_n)$ is a sequence in $C_{rc}(X, E)$ such that $\|f_n(x)\| \to 0$ for all $x \in X$, then $f_n \to 0$.

(ii) If $(f_a)$ is a net in $C_{rc}(X, E)$ such that $\|f_a(x)\| \to 0$ for all $x \in X$, then $f_a \to 0$.

**PROOF.** (i) Let $p = \|\cdot\|$ be the non-Archimedean norm of $E$ and let $W$ be a closed absolutely convex $\beta_1$-neighborhood of zero. The polar $H = W^\perp$ of $W$, in the dual space of $(C_{rc}(X, E), \beta_1)$, is $\beta_1$-equicontinuous. Let $a \in F$, $|a| > 1$. By [13, Theorems 4.14 and 4.15], we have $H^0 \subset C(a, W)$. Choose $y, \delta \neq 0$ in $F$ such that $|y| \geq \|f_1\|$, $|\delta| \leq |a|^{-1}$ and $|\delta| \cdot m_p(X) \leq |a|^{-1}$ for all $m \in \mathcal{H}$. Let 

$$A_n = \{ x \in X : \|f_n(x)\| \geq |\delta| \}.$$ 

Then, $A_n \neq \emptyset$ and so, by Theorem 5.9, there exists $n_0$ such that $m_p(A_n) < |\delta|$ for all $m \in \mathcal{H}$ and all $n \geq n_0$. Let now $n \geq n_0$. For all $m \in \mathcal{H}$, we have...
\[
\left| \sum_{n} f_n \, dm \right| \leq \| \gamma \|_{\mathcal{P}} (A_n) \leq |\alpha|^{-1}
\]
and
\[
\left| \sum_{n} f_n \, dm \right| \leq \| \delta \|_{\mathcal{P}} (X) \leq |\alpha|^{-1}.
\]
Thus, \( \left| \sum_{n} a f_n \, dm \right| \leq 1 \) for all \( m \in H \) which implies that \( a f_n \in H \subseteq \alpha W \) and so \( f_n \in W \).

(ii) The proof is analogous to that of (i).

REFERENCES

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