NORM-PRESERVING L-L INTEGRAL TRANSFORMATIONS

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ABSTRACT. In this paper we consider an L-L integral transformation G of the form $F(x) = \int_0^\infty G(x,y)f(y)dy$, where $G(x,y)$ is defined on $D = \{(x,y): x \geq 0, y \geq 0\}$ and $f(y)$ is defined on $[0,\infty)$. The following results are proved: For an L-L integral transformation G to be norm-preserving, $\int_0^\infty |G_*(x,t)|dx = 1$ for almost all $t \geq 0$ is only a necessary condition, where $G_*(x,t) = \lim_{h \rightarrow 0} \inf \frac{1}{h} \int_t^{t+h} G(x,y)dy$ for each $x \geq 0$. For certain G 's, $\int_0^\infty |G_*(x,t)|dx = 1$ for almost all $t \geq 0$ is a necessary and sufficient condition for preserving the norm of certain $f \in L$. In this paper the analogous result for sum-preserving L-L integral transformation G is proved.

KEY WORDS AND PHRASES. ℓ - ℓ method. L-L integral transformation. Absolutely continuity of integrals. Fubini-Tonelli Theorem.

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1. INTRODUCTION.

The well-known summability method defined by a ℓ - ℓ matrix $A = (a_{nk})$, mapping from ℓ into ℓ , is sum-preserving if and only if for each k , $\sum_{n \geq 1} a_{nk} = 1$. In our present study we also discuss conditions under which G defined by $\int_0^\infty G(x,y)f(y)dy$, mappings L into L , is norm-preserving or sum-preserving.

2. NOTATION.

The notation and terms used are:

The statement that f is Lebesgue integrable on $[0,\infty)$ means that for every $u > 0$, if f is Lebesgue integrable on $[0,u]$ and that $\int_0^u f(x)dx$ tends to a finite limit as $u \rightarrow \infty$.

L - the space of functions that are Lebesgue integrable on $[0,\infty)$ with norm $\|f\| = \int_0^\infty |f(x)|dx$.

D - the first quadrant of the plane, i.e., $D = \{(x,y): x \geq 0, y \geq 0\}$.

G - an integral transformation, $G: f \rightarrow F$, of the form (*) $F(x) = \int_0^\infty G(x,y)f(y)dy$, for all $x \geq 0$, where f is defined on $[0,\infty)$ and $G(x,y)$ defined on D .

$\mathcal{G}$ - the collection of all G of the form (*).

$\mathcal{GL}$ - the subcollection of $\mathcal{G}$ such that $F \in L$ whenever $f \in L$.

L^∞ - the space of functions which are measurable and essentially bounded on $[0,\infty)$ with norm $\|f\|_\infty = \text{ess - sup}_{x \geq 0} |f(x)|$

3. MAIN THEOREM

THEOREM 1. If $G \in GL$ and for every $f \in L$

$$\int_0^\infty |F(x)| dx = \int_0^\infty |f(y)| dy,$$

then for almost all $y \geq 0$,

$$\int_0^\infty |G_*(x, y)| dx = 1,$$

where $G_*(x, y) = \liminf_{h \rightarrow 0} \frac{1}{h} \int_y^{y+h} G(x, t) dt$, for each $x \geq 0$.

PROOF. Suppose that there is a set $A \subseteq \{y \geq 0\}$ satisfying $0 < mA < \infty$ such that either $\int_0^\infty |G_*(x, y)| dx > 1$ for all $y \in A$ or $\int_0^\infty |G_*(x, y)| dx < 1$ for all $y \in A$. Since $G \in GL$, for each $x \geq 0$, it follows from Theorem (T.S.T.), see [1], that for every measurable set A of finite measure, $\int_A G(x, y) dy < \infty$, without loss of generality, we can assume that A is a bounded measurable set.

Case i). Suppose that for all $y \in A$, $\int_0^\infty |G_*(x, y)| dx < 1$. Without loss of generality we assume that for each $y \in A$

$$\int_0^\infty |G_*(x, y)| dx < 1 - \epsilon,$$

where ϵ is a small positive number. Let $f(y) = \chi_A(y)$ then

$$\begin{aligned} F(x) &= \int_0^\infty G(x, y) \chi_A(y) dy \\ &= \int_A G(x, y) dy, \end{aligned}$$

and

$$\begin{aligned} \|F\| &= \int_0^\infty \left| \int_0^\infty G(x, y) \chi_A(y) dy \right| dx \\ &\leq \int_0^\infty \int_A |G(x, y)| dy dx. \end{aligned}$$

Since for each $x \geq 0$, $G_*(x, y) = G(x, y)$ for almost all $y \geq 0$, see [2, Theorem 5. P. 255], so it follows from the Fubini-Tonelli Theorem that

$$\begin{aligned} \|F\| &\leq \int_0^\infty \int_A |G(x, y)| dy dx \\ &= \int_0^\infty \int_A |G_*(x, y)| dy dx \\ &= \int_A \int_0^\infty |G_*(x, y)| dx dy \\ &\leq \int_A (1 - \epsilon) dy \\ &= (1 - \epsilon) mA \\ &< mA = \|\chi_A\|. \end{aligned}$$

Hence, for case i), G is not norm-preserving.

Case ii). Suppose that for all $y \in A$, $\int_0^\infty |G_*(x, y)| dx > 1$. Without loss of generality, we assume that for each $y \in A$

$$\int_0^\infty |G_*(x, y)| dx > 1 + \epsilon.$$

where ϵ is a small positive number. Let $f(y) = \chi_A(y)$; then $F(x) = \int_A G(x, y) dy$ for all $x \in [0, \infty)$. If $F(x) = 0$ for almost all $x \in [0, \infty)$, then

$$\|F\| = 0 < mA = \|\chi_A\| = \|f\|,$$

and we're done.

Suppose that $F(x) \neq 0$ for all x in some set with positive measure. Since $G \in GL$, so it follows from Theorem (T. S. T) by author, see [1], $G_*(x, y)$ is measur-

able on D and $\int_0^\infty |G_*(x, y)| dx < M$ for almost all $y \geq 0$, where M is a constant. Thus

$$\int_A G_*(x, y) dy = \int_A G(x, y) dy \quad \text{and} \quad \int_A \int_0^\infty |G_*(x, y)| dx dy < Mm_A < \infty, \quad \text{and} \\ \int_0^\infty \int_A |G_*(x, y)| dy dx = \int_A \int_0^\infty |G_*(x, y)| dx dy < \infty.$$

Given $1 > \varepsilon/2 > \eta > 0$, there is an $X_0 > 0$ such that

$$\int_{X_0}^\infty \int_A |G_*(x, y)| dy dx < \eta \cdot mA/2 < \varepsilon \cdot mA/4.$$

It follows that there is at least a subset $A_0 \subseteq A$, having positive measure and for all $y \in A_0$, satisfying

$$\int_{X_0}^\infty |G_*(x, y)| dx < \varepsilon/8$$

and from $\int_0^\infty |G_*(x, y)| dx > 1 + \varepsilon$ for each $y \in A$ that

$$\int_0^{X_0} |G_*(x, y)| dx > 1 + 3\varepsilon/4$$

for all $y \in A_0$. Let $E = \{(x, y) \in [0, X_0] \times A_0 : |G_*(x, y)| < \varepsilon/2^4 X_0\}$ and for any $y \in A_0$, let $E_y = \{x_1 \in [0, X_0] : (x_1, y) \in E\}$. Then $0 \leq mE_y \leq X_0$ for all $y \in A_0$.

Since

$$\int_{A_0} \int_0^{X_0} |G_*(x, y)| dx dy \leq \int_A \int_0^\infty |G_*(x, y)| dx dy,$$

so it follows from the absolute continuity of the integral that there is a $\delta > 0$ such that for every measurable set $H \subseteq [0, X_0] \times A_0$ satisfying $mH < \delta$, and

$$\iint_H |G_*(x, y)| dy dx < \eta \cdot mA_0/4.$$

If $\int_{A_0} G(x, y) dy = 0$ for almost all $x \geq 0$, then

$$\|F\| = \int_0^\infty \left| \int_0^\infty G(x, y) \chi_{A_0}(y) dy \right| dx \\ = \int_0^\infty \left| \int_{A_0} G(x, y) dy \right| dx \\ = 0 < mA_0 = \|\chi_{A_0}\|,$$

and we're done. So we suppose that $\int_{A_0} G(x, y) dy \neq 0$ for some set of $x \geq 0$ with positive measure. By the Generalization of Luzin's Theorem we can choose a closed set $F \subseteq [0, X_0] \times A_0$ such that if $H = [0, X_0] \times A_0 \setminus F$ then $mH < \delta$ and

$$\iint_H |G_*(x, y)| dx dy < \eta \cdot mA_0/4,$$

and $G_*(x, y)$ is continuous over F . It is clear that $G_*(x, y)$ is uniformly continuous on F . Thus we can have a finite number N of subsets A_i of $mA_i > 0$ of set A_0 such that $F = \bigcup_{i=1}^N [0, X_0] \times A_i$ and within each strip $[0, X_0] \times A_i$ for each $x \in [0, X_0]$ the value of $G_*(x, y)$ are close to one another. More precisely, for (x, y') , $(x, y'') \in [0, X_0] \times A_i$ and (x, y') , $(x, y'') \notin E$, $|G_*(x, y') - G_*(x, y'')| < \varepsilon/2^5 X_0$. Then for each A_i there are three sets A_i^+ , A_i^- and E_y of $x \in [0, X_0]$, such that

$$G_*(x, y) > 0, \quad \text{if } (x, y) \in F \cap (A_i^+ \times A_i)$$

$$G_*(x, y) < 0, \quad \text{if } (x, y) \in F \cap (A_i^- \times A_i),$$

and

$$|G_*(x, y)| < \varepsilon/2^4 X_0, \quad \text{if } (x, y) \in E_y \times A_i.$$

Hence, if $(x, y) \in F \cap [0, X_0] \times A_1$, then

$$\begin{aligned}
 & \int_0^{X_0} \left| \int_0^\infty G(x, y) \chi_{A_1}(y) dy \right| dx = \int_0^{X_0} \left| \int_{A_1} G(x, y) dy \right| dx \\
 &= \int_0^{X_0} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &= \int_{A_1^+} \left| \int_{A_1} G_*(x, y) dy \right| dx + \int_{A_1^-} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &+ \int_{E_y} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &= \int_{A_1^+} \int_{A_1} G_*(x, y) dy dx + \int_{A_1^-} (- \int_{A_1} G_*(x, y) dy) dx \\
 &+ \int_{E_y} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &= \int_{A_1^+} \int_{A_1^+} G_*(x, y) dx dy + \int_{A_1^-} \int_{A_1^-} (-G_*(x, y)) dy dx \\
 &+ \int_{E_y} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &= \int_{A_1} \int_{A_1^+ \cup A_1^-} |G_*(x, y)| dy dx + \int_{E_y} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &= \int_{A_1^+ \cup A_1^-} \int_{A_1} |G_*(x, y)| dy dx + \int_{E_y} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &= \int_0^{X_0} \int_{A_1} |G(x, y)| dy dx - \int_{E_y} \int_{A_1} |G(x, y)| dy dx + \int_{E_y} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &= \int_0^{X_0} \int_{A_1} |G_*(x, y)| dy dx - \left\{ \int_{E_y} \left[\int_{A_1} |G_*(x, y)| dy - \left| \int_{A_1} G_*(x, y) dy \right| \right] dx \right\} \\
 &\geq \int_{A_1} \int_0^{X_0} |G_*(x, y)| dx dy - 2 \int_{E_y} \int_{A_1} |G_*(x, y)| dy dx \\
 &> (1 + 3\epsilon/4)mA_1 - \epsilon \cdot mA_1/8 \quad (\text{since } mE_y < X_0).
 \end{aligned}$$

If $m\{H \cap [0, X_0] \times A_1\} = 0$ for some $A_1 \in \{A_1\}_1^N$, then for such an A_1 ,

$$\begin{aligned}
 \|F\| &= \int_0^\infty \left| \int_0^\infty G(x, y) \chi_{A_1}(y) dy \right| dx \\
 &= \int_0^\infty \left| \int_{A_1} G(x, y) dy \right| dx \\
 &= \int_0^{X_0} \left| \int_{A_1} G_*(x, y) dy \right| dx + \int_{X_0}^\infty \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &\geq \int_0^{X_0} \left| \int_{A_1} G_*(x, y) dy \right| dx - \int_0^{X_0} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &\quad (x, y) \in F \quad (x, y) \in H \\
 &= \int_0^{X_0} \left| \int_{A_1} G_*(x, y) dy \right| dx \\
 &> (1 + 3\epsilon/4)mA_1 - \epsilon mA_1/8 \\
 &> mA_1 = \|\chi_{A_1}\|, \text{ and we're done.}
 \end{aligned}$$

If $m\{H \cap [0, X_0] \times A_1\} \neq 0$ for all $A_1 \in \{A_1\}_1^N$, then there is at least an A_1 such that

$$\int_{H \cap [0, X_0] \times A_1} |G_*(x, y)| dy dx < (\eta \cdot mA_0/4) \cdot mA_1/mA_0,$$

and for such an A_1 ,

$$\begin{aligned}
 \|F\| &= \int_0^\infty \left| \int_0^\infty G(x,y) \chi_{A_1}(y) dy \right| dx \\
 &= \int_0^{X_0} \left| \int_0^\infty G_*(x,y) \chi_{A_1}(y) dy \right| dx + \int_{X_0}^\infty \left| \int_0^\infty G_*(x,y) \chi_{A_1}(y) dy \right| dx \\
 &\geq \int_0^{X_0} \left| \int_{A_1} G_*(x,y) dy \right| dx - \int_0^{X_0} \left| \int_{A_1} G_*(x,y) dy \right| dx \\
 &\quad (x,y) \in F \quad (x,y) \in H \\
 &\geq \int_0^{X_0} \left| \int_{A_1} G_*(x,y) dy \right| dx - \eta \cdot mA_1/4 \\
 &\quad (x,y) \in F \\
 &\geq (1 + 3\epsilon/4)mA_1 - 2\epsilon mA_1/8 \\
 &= (1 + \epsilon/2)mA_1 \\
 &> mA_1 = \|\chi_{A_1}\|.
 \end{aligned}$$

Hence, case (ii) we have proved that G is not norm-preserving and so the proof is complete.

Theorem 1 shows us that if $G \in GL$, for almost all $y \geq 0$, $\int_0^\infty |G_*(x,y)| dx = 1$ is a necessary condition for $\int_0^\infty |F(x)| dx = \int_0^\infty |f(y)| dy$ whenever $f \in L$. The next example will tell us that for almost all $y \geq 0$, $\int_0^\infty |G_*(x,y)| dx = 1$ is not a sufficient condition for $\int_0^\infty |F(x)| dx = \int_0^\infty |f(y)| dy$ for every $f \in L$.

But the following theorem will show that for certain G 's, $\int_0^\infty |G_*(x,y)| dx = 1$ is a necessary and sufficient condition for preserving the norms of certain $f \in L$.

Example. Define

$$G(x,y) = \begin{cases} -1/4x^{1/2}, & \text{if } x \in (0,1), \\ 1/2x^2, & \text{if } x \in [1,\infty), \end{cases} \quad \text{for all } y \geq 0;$$

and

$$f(y) = \begin{cases} -2/(y+1)^2, & \text{if } y \in [0,1), \\ 10/(y+1)^2, & \text{if } y \in [1,\infty). \end{cases}$$

Then

$$\begin{aligned}
 \int_0^\infty |f(y)| dy &= \int_0^1 2/(y+1)^2 dy + \int_1^\infty 10/(y+1)^2 dy \\
 &= -2(y+1)^{-1} \Big|_0^1 - 10(y+1)^{-1} \Big|_1^\infty \\
 &= -1 + 2 + 5 = 6,
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^\infty |G_*(x,y)| dx &= \int_0^1 1/4x^{1/2} dx + \int_1^\infty 1/2x^2 dx \\
 &= 2x^{1/2}/4 \Big|_0^1 - 1/2x \Big|_1^\infty \\
 &= 1/2 + 1/2 = 1.
 \end{aligned}$$

But

$$\begin{aligned}
 F(x) &= \int_0^\infty G(x,y) f(y) dy \\
 &= \begin{cases} \int_0^\infty (-1/4x^{1/2}) f(y) dy, & \text{if } x \in (0,1), \\ \int_0^\infty (1/2x^2) f(y) dy, & \text{if } x \in [1,\infty), \end{cases}
 \end{aligned}$$

where

$$\begin{aligned} - \int_0^{\infty} (1/4x^{1/2}) f(y) dy &= -1/4x^{1/2} [\int_0^1 -2/(y+1)^2 dy + \int_1^{\infty} 10/(y+1)^2 dy] \\ &= -1/4x^{1/2} [-2(-1)(y+1)^{-1} \Big|_0^1 + 10(-1)(y+1)^{-1} \Big|_1^{\infty}] \\ &= -1/4x^{1/2} [1 - 2 + 5] \\ &= -1/x^{1/2}, \quad \text{if } x \in (0,1), \end{aligned}$$

and

$$\begin{aligned} \int_0^{\infty} (1/2x^2) f(y) dy &= (1/2x^2) [\int_0^1 -2/(y+1)^2 dy + \int_1^{\infty} 10/(y+1)^2 dy] \\ &= (1/2x^2) (1 - 2 + 5) = 2/x^2, \quad \text{if } x \in [1, \infty). \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^{\infty} |F(x)| dx &= \int_0^1 1/x^{1/2} dx + \int_1^{\infty} 2/x^2 dx \\ &= 2x^{1/2} \Big|_0^1 + 2(-1)x^{-1} \Big|_1^{\infty} \\ &= 2 + 2 \\ &= 4 \neq 6 = \int_0^{\infty} |f(y)| dy. \end{aligned}$$

THEOREM 2. Suppose that $G(x,y)$ is a nonnegative function on D and $G \in GL$; then the following are equivalent;

- i) $\|F\| = \|f\|$ whenever $f \in L$ and $f(y) \geq 0$ on $[0, \infty)$;
- ii) $\|F\| = \|f\|$ whenever $f \in L$ and $f(y) \leq 0$ on $[0, \infty)$;
- iii) $\|F\| = \|f\|$ whenever $f \in L$, if $F(x) = \int_0^{\infty} G(x,y) |f(y)| dy$;
- iv) $\int_0^{\infty} G_{\star}(x,y) dx = 1$, for almost all $y \geq 0$.

PROOF. Since $\|f\| = \int_0^{\infty} |f(y)| dy$, $F(x) = \int_0^{\infty} G(x,y) f(y) dy$ and $\|F\| = \int_0^{\infty} |F(x)| dx$,

it is clear that i) is equivalent to ii). We now prove that i) is equivalent to iv). Assuming that $G(x,y) \geq 0$ on D and $f(y) \geq 0$ for all $y \in [0, \infty)$, we have $f(y)G(x,y) \geq 0$ on D . Hence

$$F(x) = \int_0^{\infty} G(x,y) f(y) dy \geq 0,$$

so

$$|F(x)| = F(x).$$

Therefore

$$\|F\| = \int_0^{\infty} |F(x)| dx = \int_0^{\infty} F(x) dx,$$

and

$$\|F\| = \int_0^{\infty} \int_0^{\infty} G(x,y) f(y) dy dx.$$

By the Fubini-Tonelli Theorem and for each $x \geq 0$, $G_{\star}(x,y) = G(x,y)$ for almost all $y \geq 0$,

$$\|F\| = \int_0^{\infty} f(y) \int_0^{\infty} G_{\star}(x,y) dx dy.$$

Hence,

$$\begin{aligned} \|F\| &= \|f\| \quad \text{if and only if} \\ \int_0^{\infty} G_{\star}(x,y) dx &= 1 \quad \text{for almost all } y \geq 0. \end{aligned}$$

Next we prove that (iii) is equivalent to (iv). Let

$$f^+ = \begin{cases} f(y), & \text{if } f(y) \geq 0 \\ 0, & \text{if } f(y) < 0 \end{cases};$$

$$f^- = \begin{cases} -f(y), & \text{if } f(y) < 0 \\ 0, & \text{if } f(y) \geq 0 \end{cases}.$$

Since $G(x, y) \geq 0$ on D , so whenever $f \in L$

$$F^+ = \int_0^\infty G(x, y) f^+(y) dy \geq 0, \quad \text{for all } x \geq 0;$$

$$F^- = \int_0^\infty G(x, y) f^-(y) dy \geq 0, \quad \text{for all } x \geq 0;$$

and

$$f(y) = f^+ - f^-,$$

if $F(x) = \int_0^\infty G(x, y) |f(y)| dy$, then

$$|F(x)| = F^+ + F^-.$$

It follows from (i) that

$$\begin{aligned} \|F(x)\| &= \int_0^\infty |F(x)| dx \\ &= \int_0^\infty |f(y)| dy \end{aligned}$$

if and only if

$$\int_0^\infty G_*(x, y) dx = 1 \quad \text{for almost all } y \geq 0.$$

We are also interested in the analogous sum-preserving question for L-L integral transformations, viz., when is $\int_0^\infty F(x) dx = \int_0^\infty f(y) dy$ whenever $f \in L$?

Next we give the definition of sum-preserving for L-L integral transformations and a result concerning it.

DEFINITION. The integral transformation $G \in GL$ is said to be sum-preserving if and only if

$$\int_0^\infty F(x) dx = \int_0^\infty f(y) dy$$

for all $f(y) \in L$, where $F(x) = \int_0^\infty G(x, y) f(y) dy$.

COROLLARY. Suppose that $G(x, y)$ is a nonnegative function on D and $G \in GL$; then G is a sum-preserving transformation whenever $f \in L$ if and only if $\int_0^\infty G_*(x, y) dx = 1$ for almost all $y \geq 0$.

PROOF. Since $f \in L$, $f = f^+ - f^-$, where

$$f^+ = \begin{cases} f(y), & \text{if } f(y) \geq 0 \\ 0, & \text{if } f(y) < 0 \end{cases},$$

$$f^- = \begin{cases} -f(y), & \text{if } f(y) < 0 \\ 0, & \text{if } f(y) \geq 0 \end{cases},$$

and

$$\int_0^\infty f(y) dy = \int_0^\infty f^+ dy - \int_0^\infty f^- dy.$$

Then

$$\begin{aligned} F(x) &= \int_0^\infty G(x,y) f(y) dy \\ &= \int_0^\infty G(x,y) [f^+ - f^-] dy \\ &= \int_0^\infty G(x,y) f^+(y) dy - \int_0^\infty G(x,y) f^-(y) dy \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty F(x) dx &= \int_0^\infty \int_0^\infty G(x,y) f^+(y) dy - \int_0^\infty \int_0^\infty G(x,y) f^-(y) dy dx \\ &= \int_0^\infty \int_0^\infty G(x,y) f^+(y) dy dx - \int_0^\infty \int_0^\infty G(x,y) f^-(y) dy dx . \end{aligned}$$

By the Fubini-Tonelli Theorem

$$\int_0^\infty \int_0^\infty G(x,y) f^+(y) dy dx = \int_0^\infty f^+(y) \int_0^\infty G_*(x,y) dx dy ;$$

and

$$\int_0^\infty \int_0^\infty G(x,y) f^-(y) dy dx = \int_0^\infty f^-(y) \int_0^\infty G_*(x,y) dx dy .$$

Thus

$$\int_0^\infty \int_0^\infty G(x,y) f^+(y) dy dx = \int_0^\infty f^+(y) dy ;$$

and

$$\int_0^\infty \int_0^\infty G(x,y) f^-(y) dy dx = \int_0^\infty f^-(y) dy$$

if and only if

$$\int_0^\infty G_*(x,y) dx = 1 \quad \text{for almost all } y \geq 0 .$$

Therefore

$$\int_0^\infty F(x) dx = \int_0^\infty f^+ dy - \int_0^\infty f^- dy$$

if and only if

$$\int_0^\infty G_*(x,y) dx = 1 \quad \text{for almost all } y \geq 0 ,$$

i.e.,

$$\int_0^\infty F(x) dx = \int_0^\infty f(y) dy$$

if and only if

$$\int_0^\infty G_*(x,y) dx = 1 \quad \text{for almost all } y \geq 0 .$$

The proof is completed.

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