ON CENTER-LIKE ELEMENTS IN RINGS

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ABSTRACT. In a paper with a similar title Herstein has considered the structure of prime rings which contain an element a which satisfies either \([a, x]^n = 0\) or is in the center of \(R\) for each \(x\) in \(R\). We consider the structure of rings which contain an element a which satisfies powers of certain higher commutators. The two types which we consider are (1) \([[[a, x_1], x_2],...,x_m]^n = 0\) or is in the center of \(R\) for all \(x_1, x_2,...,x_m\) in \(R\) and (2) \([a, [[x_1, [x_2,...,[x_{m-1},x_m],...]]]^n = 0\) for all \(x_1, x_2,...,x_m\) in \(R\). We obtain results similar to those obtained by Herstein; however, in some cases we must strengthen the hypotheses.

Also we consider a third type (3) \((ax^m - x^na)^k = 0\) for all \(x\) in \(R\) and extend results of Herstein and Giambruno.

KEY WORDS AND PHRASES. Prime and semiprime rings, primitive and semiprimitive rings, rings with involution, commutativity theorems.

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1. INTRODUCTION.

The definition of the center \(Z\) of a ring \(R\) has recently been generalized in several papers. Herstein [1, Theorem 2] showed that an element \(a\) of a prime ring \(R\) is central if and only if \([a, u]^n = 0\) for all \(u \in U\) where \(U\) is a nonzero two-sided ideal in \(R\). We generalize this result in two directions. First, we show that (1) if \(R\) is prime and \([[[a, u_1], u_2],...,u_m]^n = 0\) for all \(u_1, u_2,...,u_m \in U\), then \(a \in Z\). From (1) it follows easily that a semiprime ring satisfying the Lie nilpotent identity \([...[x_1, x_2],...,x_m]^n = 0\) for all \(x_1, x_2,...,x_m\) in \(R\) is commutative [2, p. 230]. We also conclude from (1) two commutivity theorems which generalize two well-known theorems due to Kaplansky [2, p. 219] and Herstein [3].

Second, we prove that if \([a, [u_1, [u_2,...,[u_{m-1},u_m],...]]^n = 0\) for all \(u_1, u_2,...,u_m\) in \(U\), then \(a \in Z\) if either \(R\) is semisimple and \(U\) is essential, or \(R\) is prime with \(Z\) infinite and \(n\) fixed.

Herstein [1, Theorem 4] proves that if \(R\) is prime, \(a \not\in Z\), and \([a, x]^n \in Z\) for all \(x \in R\), then \(R\) is an order in a 4-dimensional simple algebra. We show that the
same result holds if \( \ldots [[[a,x_1],x_2], \ldots, x_m] \in Z \) for all \( x_1, x_2, \ldots, x_m \in R \).

In another attempt to generalize the structure of the center \( Z \) of a ring \( R \) without nonzero nil ideals Herstein [4] proved that the subring \( T = \{ a \in R : a^n(x, a) = x^n(a, a) \text{ for all } x \in R \} = Z \). This theorem was generalized by Ciambruno [5], who showed that the set \( G = \{ a \in R : a^m(x, a) = x^n(a, a) \text{ for all } x \in R \} = Z \). In an attempt to generalize these results, we show that \( G = \{ a \in R : (a^m(x, a) - x^n(a, a))^k = 0 \text{ for all } x \in R \} = Z \) if \( R \) is semiprimitive and \( 2R \neq 0 \).

Throughout this paper \( R \) is an associative ring with \( 1 \) and \( Z \) denotes the center of \( R \). Moreover, \([a,x] = ax - xa\) and if \( X \) is a subset of \( R \), then \( \mathcal{L}(X) = \{ r \in R : rx = 0 \text{ for all } x \in X \} \).

2. MAIN RESULTS. We begin this section with a lemma which will be useful in the sequel.

**LEMMA 1.** Let \( R \) be a ring, \( U \) an ideal of \( R \), and \( a \in R \). If \( [[a,ux],u] \in U \) for all \( u, x \in U \), then \( [a,u] \in U \).

**PROOF.** Let \( u \in U \). Since \( U \) is an ideal we obtain

\[
0 = [[a,au],u] = [a[a,u],u] + [a,u]u
\]

However the first term is zero. Hence \( [a,u] \in U \).

**THEOREM 1.** Let \( R \) be a prime ring and \( U \neq 0 \) an ideal of \( R \). If \( a \in R \) is such that for fixed positive integers \( m \) and \( n \), \( [[[a,u_1],u_2], \ldots, u_m] = 0 \) for all \( u_1, u_2, \ldots, u_m \in U \), then \( a \in Z \).

**PROOF.** The proof goes by induction on \( m \). The result is true for \( m = 1 \) by Herstein's theorem [1, Theorem 2].

Assume the result is true for \( k < m \) and suppose that \( [[[a,u_1],u_2], \ldots, u_m] = 0 \) for all \( u_1, u_2, \ldots, u_m \in U \). Set \( b = [a,u_1] \). Then by assumption

\( [[[b,u_2],u_3], \ldots, u_m] = 0 \)

for all \( u_2, u_3, \ldots, u_m \in U \). Hence \( b \in Z \) by the induction hypothesis. By applying Lemma 1 we obtain that \( [a,u] = 0 \) for all \( u \in U \). Therefore \( a \in Z \) by Herstein's aforementioned theorem.

As a consequence of Theorem 1, we get the following two corollaries which generalize for prime rings two well-known theorems due to Kaplansky [2, p. 219] and Herstein [3].

**COROLLARY 1.** Let \( R \) be a prime ring and \( U \neq 0 \) an ideal of \( R \). If for every \( a \in R \) there exists three natural numbers \( k(a) \), \( m(a) \), and \( n(a) \) such that

\( [[[a^k(u_1),u_2], \ldots, u_m] = 0 \)

where \( u_1, u_2, \ldots, u_m \in U \), then \( R \) is commutative.

**PROOF.** Evident.

**COROLLARY 2.** Let \( R \) be a prime ring and \( U \neq 0 \) and ideal of \( R \). If for every \( a \in R \), there exists two natural numbers \( m(a) \) and a polynomial \( p_a(x) \) with integer coefficients such that

\( [[[a - a^2p_a(u_1),u_2], \ldots, u_m] = 0 \)

where \( u_1, u_2, \ldots, u_m \in U \), then \( R \) is commutative.
PROOF. Evident.

Also as a corollary we obtain a result from [2, p. 230].

COROLLARY 3. If \( R \) is a semiprime ring satisfying the Lie nilpotent identity 
\[ \ldots[[x_1,x_2],\ldots,x_n] = 0, \]  
then \( R \) is commutative.

PROOF. Evident.

The next theorem generalizes a theorem of Herstein [1, Theorem 3].

THEOREM 2. Let \( R \) be a prime ring with center \( Z \) and let \( a \in R \), \( a \notin Z \) be such that 
\[ \ldots[[a,u_1],[u_2],\ldots,[u_m]] \in Z \]  
for all \( u_1,u_2,\ldots,u_m \in U \) where \( U \neq 0 \) is an ideal of \( R \). Then \( R \) is an order in a 4-dimensional simple algebra.

PROOF. If \( \ldots[[a,u_1],[u_2],\ldots,[u_m]] \in Z \) for all \( u_1,u_2,\ldots,u_m \in U \), then \( a \in Z \) by Theorem 2. Hence there exists \( v_1,v_2,\ldots,v_m \in U \) such that 
\[ b = \ldots[[a,v_1],[v_2],\ldots,v_m] \notin Z. \]
However by hypothesis \( [b,u_m] \in Z \) for all \( u_m \in U \). Ergo, \( R \) is an order in a 4-dimensional simple algebra by Herstein [1, Theorem 3].

We now generalize Herstein's Theorem 2 in [1] in another direction. Let \( U \neq 0 \) be an ideal of \( R \), \( a \in R \), \( m \) fixed in \( Z^+ \). If 
\[ [a,[u_1],[u_2],\ldots,[u_m]] = 0 \]
for all \( u_1,u_2,\ldots,u_m \in U \) (Condition A) then we shall prove that \( a \in Z \) in the following two cases:

(i) \( R \) is semiprimitive and \( U \) is an ideal such that \( \ell(U) = 0 \) (Theorem 3), or

(ii) \( R \) is prime, \( U \) is an ideal, \( Z \) is infinite, and \( n \) is fixed (Theorem 4).

First we prove a lemma:

LEMMA 2. Let \( R \) be a primitive ring, \( U \neq 0 \) an ideal of \( R \), and \( a \in R \) satisfying condition (A). Then \( a \in Z \).

PROOF. (a) If \( R \) is a division ring, then \( [a,[x_1],[x_2],\ldots,[x_n]] = 0 \) for all \( x_1,x_2,\ldots,x_n \in R \). Hence \( a \in Z \) by a result of Smiley [6].

(b) If \( R \) is primitive, then it has a faithful irreducible \( R \)-module \( V \) which is also faithful and irreducible as a \( U \)-module. By the Density theorem \( U \) acts densely on \( V \) as a vector space over a division ring \( D \). If \( \dim_D V = 1 \), then \( R = D \) and the result follows from (a). So let \( \dim_D V > 1 \).

Suppose that there exists a nonzero vector \( v \in V \) such that \( v \) and \( va \) are linearly independent over \( D \). Since \( U \) acts densely on \( V \) there exists \( u_1,u_2 \in U \) such that \( vu_1 = v \), \( (va)u_1 = v \), \( vu_2 = 0 \), and \( (va)u_2 = va \). Thus 
\[ v[a,[u_1],[u_2],\ldots,[u_1,u_2]] = v \]
and so \( v[a,[u_1],[u_2],\ldots,[u_1,u_2]]] = v \). But, by the hypothesis, the expression on the left is zero, which gives that \( v = 0 \), contrary to our assumption. Thus for every \( v \in V \), \( va = \lambda(v)v \), where \( \lambda(v) \in D \). It follows easily from this that, in fact, \( \lambda(v) \) does not depend on \( v \), hence \( va = \lambda v \) for all \( v \in V \). So, if \( x \in R \), then \( (vx)a = \lambda vx \) and \( (va)x = \lambda(v)x \). Hence \( v(xa - xa) = 0 \) for all \( v \in V \). Since \( R \) acts faithfully on \( V \) we have \( ax - xa = 0 \) for all \( x \in R \), and so \( a \in Z \).
THEOREM 3. If $R$ is a semiprimitive ring, $U \neq 0$ an ideal of $R$ with $\ell(U) = 0$, and $a \in R$ which satisfies condition $A$, then $a \in Z$.

PROOF. Since $\ell(U) = 0$, $U$ is an essential ideal of $R$. Hence it can easily be shown that $\pi(P) = P$ primitive ideal such that $P \neq U = 0$. Hence $R$ is the subdirect product of $R/P$ where $P \neq U$. It follows from Lemma 2 that $a$ is in the center of each $R/P$. Therefore $a \in Z$.

THEOREM 4. Let $R$ be prime with $Z$ infinite, $U \neq 0$ an ideal of $R$, and $a \in R$ which satisfies condition $A$, then $a \in Z$.

PROOF. Let $C$ be the extended centroid of $R$. Then $C \supset Z$ and because $Z$ is infinite condition $A$ carries over to the prime ring $S = RC$ and its ideal $V = UC$. If $a \notin Z$ then $R$ satisfies a nontrivial generalized polynomial identity $[a, [x_1, x_2, \ldots, [x_m - x_n, x_n, \ldots]]] = 0$ for $x_1, x_2, \ldots, x_m \in R$. Hence $S = RC$ is primitive by Martindale's theorem. Since $V = UC$ is an ideal of $S$ which satisfies condition $A$, we have that $a \in Z(S)$ by Lemma 2. Hence $a \in Z$.

Question 1: In Theorem 4 is the hypothesis that $Z$ be infinite necessary? Note that in Theorem 1 it was not necessary.

We finish our paper with a partial generalization of the results in [5] and [4]. Let $a$ be an element of the ring $R$ such that for all $u \in U$, a nonzero ideal of $R$, we have

$$(au^m(u) - u^m(a))k(u) = 0 \quad (\text{Condition B})$$

and let $\overline{a} = \{a \in R: (ax^m(x) - x^m(a))k(x) = 0 \text{ for all } x \text{ in } R\}$. It is clear that $\overline{a} \supset G \supset T \supset Z$.

THEOREM 5. If $R$ is a ring satisfying condition (B) with $2R \neq 0$, then either

1. $R$ is semiprimitive with $k(U) = 0$ or
2. $R$ is prime with infinite center with fixed integers $m$, $n$, and $k$.

Then $a \in Z$.

PROOF. By using the same technique of proof as that in Theorems 3 and 4, it is enough to prove the result in the primitive case.

Let $V$ and $D$ be as in the proof of Lemma 2. If $\dim_D V = 1$, i.e., $R$ is a division ring, we get that for all $x \in R$, $ax^m(x) - x^m(a) = 0$. Hence by a result of Giambruno [8] $a \in Z$. Thus let $\dim_D V > 1$. If $0 \neq v \in V$, then the vectors \{v, va, va^2\} are linearly dependent. Indeed, if they were linearly independent, then by the density theorem, there is $u \in U$ such that $vu = v$, $(va)u = v$, and $(va^2)u = 0$.

Thus we get $v(au^m(u) - u^m(a))k(u) = v$ if $k(u)$ is even and equals $v - va$ if $K(u)$ is odd. But $v(au^m(u) - u^m(a))k(u) = 0$ so we get a contradiction in both cases.

Assume that \{v, va\} are linearly independent, then $va^2 = \lambda v + \mu va$ where $\lambda, \mu \in D$. If $\lambda \neq 0$, then by the density theorem there is $w \in U$ such that $vw = v$ and $(va)w = 0$. So $v(aw^m(w) - w^m(a)k(w) = s\lambda v$ where $s = s(k)$. Contradiction.

However, if $\lambda = 0$, i.e., $va^2 = \mu va$, then there is $y \in U$ such that $vy = v$ and $(va)y = \alpha v$ where $0 \neq \alpha \in D$, $\alpha \neq 0$ (because $2R \neq 0$ implies $D \neq Z_2$). Thus $v(ay^m(y) - y^m(a)k(y) = \beta v - \gamma(va) = 0$ where $0 \neq \beta = \beta(k) \in D$ and $\gamma = \gamma(k) \in D$. Contradiction. Therefore \{v, va\} are linearly dependent. The same argument as used in the proof of Lemma 2 shows that $a \in Z$. This completes the proof.

REMARKS: 1) It is of interest to study all the above theorems for rings with
involution "s" by applying the same conditions on the set of symmetric elements. For example, it is natural to ask: If \([a, s_1, \ldots, s_n]^n \in Z\) for all \(s_1, s_2, \ldots, s_n \in S\) and \(a \notin Z\), then what about \(R\)? It was shown by Fahmy \(9\) and Giambruno \(8\), that if \([s_1, s_2, \ldots, s_n]^n \in Z\), then \(\dim_2 R \leq 16\).

2) A second direction in which one may try to extend the above theorems is to generalize the cohypercenter introduced by Chacron in \(10\), i.e., to study the set \(\{a \in R : [a, x - x^2 p(x)]^n(x) = 0\} \text{ for all } x \in R\) where \(p(x)\) is a polynomial with integral coefficients.

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