Let $\tilde{M}(\tilde{\Omega}, \tilde{\eta}, \tilde{\zeta}, \tilde{g})$ be a pseudo-Riemannian manifold of signature $(n+1, n)$. One defines on $\tilde{M}$ an almost cosymplectic para $f$-structure and proves that a manifold $\tilde{M}$ endowed with such a structure is $\xi$-Ricci flat and is foliated by minimal hypersurfaces normal to $\xi$, which are of Otsuki's type. Further one considers on $\tilde{M}$ a $2(n-1)$-dimensional involutive distribution $P^\perp$ and a recurrent vector field $\tilde{V}$. It is proved that the maximal integral manifold $\tilde{M}$ of $P^\perp$ has $\tilde{V}$ as the mean curvature vector (up to $1/(2(n-1))$). If the complimentary orthogonal distribution $\tilde{P}$ of $P^\perp$ is also involutive, then the whole manifold $\tilde{M}$ is foliate. Different other properties regarding the vector field $\tilde{V}$ are discussed.

KEY WORDS AND PHRASES. Pseudo-Riemannian manifold, cosymplectic manifold, para $f$-structure, minimal hypersurface.

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1. INTRODUCTION.

Recently, many papers were devoted to $f$-structures or para $f$-structures (Ishichara and Yano [1]; Kiritchenko [2]; Yano and Kon [3]; Sinha [4]).

In this paper we consider a $C^\infty$-pseudo-Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $2n+1$ and of inertia index $n+1$ and such that the $(1,1)$-tensor field $f$ coincides with the para-complex operator $U$ (Libermann [5]) of square $+1$. Furthermore we suppose that $\tilde{M}$ is equipped with a triple $(\tilde{\Omega}, \tilde{\eta}, \xi)$ where

\begin{align*}
1^o. \tilde{\Omega} & \text{ is a canonical 2-form of rank } 2n \text{ exchangeable with the para-Hermitian component } \tilde{g}_{\tilde{\eta}} \text{ of the metric tensor } \tilde{g}; \\
2^o. \tilde{\eta} & \text{ is a canonical 1-form such that } (\tilde{\Omega})^{\tilde{\eta}} \tilde{\eta} \neq 0; \\
3^o. \xi & \text{ is the canonical vector field such that }
\begin{align*}
\tilde{\eta}(\xi) &= 1, \quad \xi \tilde{g} = 0, \quad U\xi = 0, \\
\tilde{d}\tilde{\eta} &= 0, \quad \tilde{g}(\tilde{\zeta}_{\xi}, \tilde{\zeta}_{\xi}) = \tilde{g}((\tilde{\zeta}_{\xi}, \xi, \xi).
\end{align*}
\end{align*}

(1.1)

In (1.1) $\tilde{V}$ is the covariant differential operator on $\tilde{M}$ and $\tilde{\zeta}, \tilde{\zeta}'$ are any vector fields on $\tilde{M}$.

If the above conditions are satisfied, we say that $\tilde{M}$ is endowed with an
almost cosymplectic para f-structure (abr. a.c.p. f-structure). In this case \( \tilde{\mathcal{M}} \) is called an a.c.p. f-manifold.

The differential distribution \( D_\eta = \{ Z \in \tilde{T}\mathcal{M}, \tilde{\eta}(Z) = 0 \} \) on \( \tilde{\mathcal{M}} \) is involutive and is called horizontal.

It is proved that an a.c.p. f-manifold is always \( \xi \)-Ricci flat and that it is foliated by minimal hypersurfaces \( \tilde{M}_\eta \), tangent to \( D_\eta \), which are of Otsuki's type (Otsuki [6]).

Suppose now that \( D \) and \( D^\perp \) are two complementary orthogonal differential distributions in \( D_\eta \) and \( \tilde{\mathcal{Z}} \) is a vector field in \( D \). If one has

\[
\tilde{\mathcal{V}} = \tilde{\mathcal{U}} \circ \tilde{\mathcal{X}} + \tilde{\mathcal{X}} \circ \tilde{\mathcal{U}} \circ \tilde{\mathcal{X}}^2 + \tilde{\mathcal{V}} \circ \xi
\]

for all \( \tilde{\mathcal{X}} \in D, \tilde{\mathcal{X}}^2 \in D^\perp \), and \( \tilde{\mathcal{U}}, \tilde{\mathcal{V}} \in \Lambda^1(\tilde{\mathcal{M}}) \), we say that \( D \) is contact covariant decomposable (abr. c.c.d.). Let \( \tilde{P} \) be a c.c.d. hyperbolic 2-plane of \( D_\eta \). If the dual forms of two null vector fields which define \( \tilde{P} \) form an exterior recurrent pairing (Rosca [7]; Morvan and Rosca [8]), we say that the manifold \( \tilde{\mathcal{M}} \) admits a strict c.c.d. hyperbolic 2-plane.

With the pairing \( (\tilde{P}, \tilde{P}^\perp) \) are associated a vector field \( \tilde{\mathcal{V}} \in \tilde{P} \) (called the recurrence vector field) and two vector fields \( \tilde{\mathcal{X}}_n, \tilde{\mathcal{X}}_{2n} \in \tilde{P}^\perp \) (called the distinguished vector fields).

In the present paper the following properties are proved:

(i) The \( 2(n-1) \)-distribution \( \tilde{P}^\perp \) is always involutive and the mean curvature vector field of its maximal integral manifold \( \tilde{M}^\perp \) is (up to a constant factor) equal to the induced vector field of \( \tilde{\mathcal{V}} \).

(ii) The simple unit form \( \tilde{\mathcal{V}} \) of \( \tilde{P}^\perp \) is exterior recurrent and \( \tilde{\mathcal{V}} \) is a characteristic vector field of \( \tilde{\mathcal{V}} \).

(iii) The necessary and sufficient condition for \( \tilde{M}^\perp \) to be quasi-minimal (Chen [9]) is that \( \tilde{\mathcal{X}}_n \) or \( \tilde{\mathcal{X}}_{2n} \) be a null vector field, and the necessary and sufficient condition for \( \tilde{M}^\perp \) to be minimal is that both \( \tilde{\mathcal{X}}_n \) and \( \tilde{\mathcal{X}}_{2n} \) be null vector fields.

(iv) If \( \tilde{M}^\perp \) is minimal, then the distribution \( \tilde{P} \) is also involutive and the integral surfaces of \( \tilde{P} \) are totally geodesic in \( \tilde{M}_\eta \), which in this case is foliate.

(v) Both vector fields \( \tilde{\mathcal{X}}_n \) and \( \tilde{\mathcal{X}}_{2n} \) are \( U \)-geodesic directions on \( \tilde{M}^\perp \).

2. ALMOST COSYMPLECTIC PARA f-MANIFOLD \( \tilde{\mathcal{M}}(U, \tilde{\eta}, \tilde{\n}, \tilde{\xi}) \).

Let \( (\tilde{\mathcal{M}}, \tilde{\mathcal{G}}) \) be a \( C^\infty \)-pseudo-Riemannian manifold of dimension \( 2n+1 \) and of inertia index \( n+1 \).

If \( \tilde{\mathcal{M}} \) is equipped with a non-zero tensor field \( f \) of type \((1,1)\) of constant rank and such that

\[
f(t^2-1) = 0
\]

(\( t \) is the identity tensor), then \( f \) is called a para f-structure (Sinha [4]).

In the following we suppose that \( f \) coincides with the para-complex operator \( U \) (Libermann [5]). In addition, we suppose that \( \tilde{\mathcal{M}} \) is equipped with the triple \((\tilde{\n}, \tilde{\xi}) \) where:

1°. \( \tilde{\n} \) is a canonical 2-form of rank \( 2n \) exchangeable with the para-Hermitian
component $\tilde{g}_\eta$ of the metric tensor $\tilde{g}$ (Buchner and Rosca [10]).

2. $\tilde{\eta}$ is a canonical 1-form such that $(\wedge^2 \tilde{\eta}) \eta \neq 0$ everywhere.

3. $\xi$ is the canonical vector field such that

$$\tilde{\eta}(\xi) = 1, \ i_\xi \tilde{\eta} = 0; \ i: \text{interior product.}$$

(2.2)

If one has

$$U^2 - 1 = -\tilde{\eta} \otimes \xi \Rightarrow U \xi = 0,$$

(2.3)

$$d \tilde{\eta} = 0,$$

(2.4)

$$\tilde{g}(\nu_2 \xi, \zeta) = \tilde{g}(\nu_2 \xi, \zeta)$$

(2.5)

where $\tilde{\nu}$ is the covariant differential operator on $\tilde{\mathcal{M}}$ and $\tilde{\zeta}, \tilde{\zeta}'$ are vector fields in $\tilde{\mathcal{M}}$ we say that $(U, \tilde{\eta}, \xi, \tilde{g})$ defines on $\tilde{\mathcal{M}}$ an almost cosympletic para f-structure (abr. a.c.p. f-structure) and $\tilde{\mathcal{M}}(U, \tilde{\eta}, \xi, \tilde{g})$ is called an a.c.p. f-manifold.

The differentiable distribution $D_\eta$ on $\tilde{\mathcal{M}}$ defined by

$$D_\eta = \{ X \in T\tilde{\mathcal{M}}, \tilde{\eta}(X) = 0 \}$$

is called horizontal.

It is worthwhile to note that equations (2.3), (2.4) and (2.5) show that on $\tilde{\mathcal{M}}$ the triple $(U, \tilde{\eta}, \xi)$ defines an almost paracontact structure (Sinha [4]), $D_\eta$ defines a $(2n)$-foliation, and $\xi$ is a gradient.

Let $\tilde{W} = \text{vect} \{ h_a, h_a^*, h_0 ; a = 1, \ldots, n, a^* = a + n \}$ be a local field of Witt frames (Vranceanu and Rosca [11]).

One has (Libermann [5]):

$$\tilde{\lambda}_a = h_a, \tilde{\lambda}_a^* = -h_a^*, \ U \xi = 0$$

(2.6)

and at each point $\tilde{\mathcal{P}} \in \tilde{\mathcal{M}}$ one has the splitting

$$(D_\eta)_{\tilde{\mathcal{P}}} = \tilde{S}_\eta \oplus \tilde{S}_\eta^*$$

(2.7)

where $\tilde{S}_\eta$ and $\tilde{S}_\eta^*$ are two self-orthogonal vector n-spaces spanned by $\{ h_a \}$ and $\{ h_a^* \}$ respectively.

Since the null vector fields $h_a$ and $h_a^*$ are normed, one may write

$$\tilde{g}(h_a, h_A) = 0, \ \tilde{g}(h_a^*, h_B) = 0,$$

$$\tilde{g}(h_a, h_a^*) = 1, \ \tilde{g}(\xi, \xi) = 1$$

(2.8)

where $A, B = 0, 1, \ldots, 2n; A \neq a^*, B \neq a$.

Now let $\{ \omega^A \}$ be the dual basis of $\tilde{W}$ and $\omega^A = \omega^A_{\ BC} (\omega^C_B \in C^\infty(\tilde{\mathcal{M}}))$ be the connection forms on $\tilde{\mathcal{M}}$. Then the line element $dp$ of $\tilde{\mathcal{M}}$ (dp is a canonical vectorial 1-form) and the connection equations are expressed by

$$dp = \omega^A \otimes h_A + \omega^A_{\ BC} \otimes h_B + \psi \otimes \xi$$

(2.9)

and

$$\psi h_A = \omega^A \otimes h_B$$

where $\tilde{\psi}$ is the covariant differentiation operator on $\tilde{\mathcal{M}}$. By (2.8) and (2.10) one finds
\[
\begin{align*}
\gamma^a = \gamma^a = 0, & \quad \gamma^a = 0, \\
\gamma^a + \gamma^b = 0, & \quad \gamma^a + \gamma^b = 0, \quad \gamma^a = 0, \\
\gamma^a + \gamma^b = 0, & \quad \gamma^a + \gamma^b = 0,
\end{align*}
\]

and the structure equations (E. Cartan) may be written in the following symbolic form:
\[
d\xi = -\Theta^\omega
\]

and
\[
d\theta = -\Theta^\omega + \Theta
\]

where \( \Theta \equiv \Theta^\omega_B \) are the curvature 2-forms.

Further taking into account (2.4), we may set
\[
\gamma^a = \omega, \quad \gamma^a = -\gamma^a.
\]

Now by means of (2.10), (2.11) and (2.14) one gets
\[
\gamma^a = \omega^a \otimes h_a - \omega^a \otimes h_a^a.
\]

In addition it follows from (2.15) that,
\[
\gamma^\xi = 0
\]

which proves that \( \xi \) is a geodesic direction.

From (2.9) and (2.8) one gets
\[
\gamma^a = <d\omega, d\omega> = 2 \sum \gamma^a \otimes \gamma^a + \gamma \otimes \gamma
\]

where \( \gamma_x = 2 \sum \omega^a \otimes \omega^a \) is the para-Hermitian (Buchner and Rosca [10]) component of the metric tensor \( \gamma \).

The 2-form \( \gamma \) which is exchangeable with \( \gamma_x \) is expressed by
\[
\gamma^a = \omega^a \otimes \omega^a.
\]

Using (2.15), we can find the following expression of the quadratic differential form \( <\gamma^\xi, \gamma^\xi> \):
\[
<\gamma^\xi, \gamma^\xi> = -2 \sum \gamma^a \otimes \gamma^a = -\gamma_x.
\]

Denote by
\[
\gamma^a = \omega^1 \Lambda ... \Lambda \omega^n
\]

the simple unit form corresponding to \( D \). One may write the volume element \( \omega \) of \( M \) as
\[
\omega = \gamma \Lambda \gamma.
\]

If \( L_Z \) means the Lie derivative in the direction \( Z \), then by a simple argument one can find
\[
L_Z \omega = d\omega = (\text{div } \xi) \omega.
\]

Using (2.12) and (2.13), one gets
\[
d\omega = 0, \quad \text{and this yields}
\]
\[
\text{div } \xi = 0.
\]

But on a Riemannian or pseudo-Riemannian manifold the following Yano integral
\[
\int M
\]
formula holds (Yano and Kon [12]):
\[
\text{div}(\nabla Z) - \text{div}(\text{div} \ Z) = \text{Ric}(Z) + \sum_{A,B} g(\nabla e_A Z, e_B) g(e_A, \nabla e_B Z) - (\text{div} \ Z)^2.
\]
In (2.24) Z, Ric and \{e_A\} are arbitrary vector fields on M, the Ricci tensor of \nabla and a vectorial basis respectively.

Continuing the consideration, one finds (2.24) and (2.15) by means of (2.5).

Taking into account (2.8), a short computation gives \( \text{Ric}(\xi) = 2n \).

Hence \( \nabla \) is Ricci constant in the direction of the structure vector \( \xi \) (or \( \xi \)-Ricci constant).

On the other hand, by means of (2.19) and (2.4) one sees that \( \nabla \) is co-closed, i.e. \( \delta \nabla = 0 \). Hence since \( d \xi = 0 \), it follows that \( \nabla \) is harmonic. Then if we denote by \( M_\eta \) the leaf of \( D_\eta \), it follows from the theorem of Tachibana [13] that \( M_\eta \) is minimal. This property can also be verified by a direct computation.

Since the induced value \( \nabla |_{M_\eta} \) of the almost sympletic form \( \nabla \) is also almost symplectic, the submanifold \( M_\eta \) is an example of a minimal submanifold having an almost symplectic structure \( \nabla \).

If \( \nabla \) is endowed with a para co-Kaehlerian structure (Buchner and Rosca [10]), then \( \nabla \) is a symplectic form.

Denote now by \( \nabla \) the induced value on \( M_\eta \) of the quadratic differential form given by (2.19). Since \( \xi \) is normal to \( M_\eta \), then, as is known, \( \nabla \) represents the third fundamental form of \( M_\eta \).

Thus according to (2.19) \( \nabla \) is conformal to the metric of \( M_\eta \). Taking into account of the para-Hermitian form of \( \nabla \) and (2.15), it is easy to see that \( M_\eta \) possesses principal curvatures equal to \(+1\) and principal curvatures equal to \(-1\). Therefore referring to Otsuki [6], we may say that \( M_\eta \) is a minimal hypersurface of Otsuki's type.

**THEOREM 1.** Let \( \nabla(U, u, \eta, \xi, g) \) be a pseudo-Riemannian manifold endowed with an a.c.p. f-structure. Such a manifold is \( \xi \)-Ricci constant and is foliated by minimal hypersurfaces \( M_\eta \) of Otsuki's type which are orthogonal to the structure vector field \( \xi \).

### 3. CONTACT COVARIANT DECOMPOSABLE DISTRIBUTIONS ON \( \nabla(U, u, \eta, \xi, g) \).

Referring to the definition given by Rosca [7] we give now the following

**DEFINITION.** Let \( \nabla \) be an odd-dimensional \( C^\infty \)-Riemannian (resp. \( C^\infty \)-pseudo-Riemannian) manifold equipped with an almost contact (resp. almost para contact) structure defined by a structure l-form \( \eta \) and a structure vector field \( \xi \). Let \( D_\eta \), \( D \) and \( \nabla \) be the horizontal distribution defined by \( \eta = 0 \), a differentiable distribution of \( D_\eta \) and the covariant differentiation operator on \( \nabla \). Let \( D^A \) be the complementary orthogonal distribution of \( D \) in \( D_\eta \) and \( \nabla \) be a vector field of \( D \). Then if one has
\[
\nabla \nabla \nabla = u \otimes \nabla u + u^A \otimes \nabla u^A + \nabla \otimes \xi
\]
where \( \nabla \in D, \nabla u^A \in D^A \) and \( u, u^A, \nabla \in \Lambda^1(\nabla) \), we say that the distribution \( D \) is contact covariant decomposable (abr. c.c.d.).

As is known, the null vectorial basis \( \{h_a, h_{a^*}\} \) of \( D_\eta \) admits the orthogonal
decomposition
\[ D^n = P_1 \ldots P_a \ldots P_n \] (3.2)
where \( P_a = (h_a, h_a^*) \) is a hyperbolic 2-plane.

We say that the a.c.p. f-manifold \( \mathcal{M}(U, \omega, \eta, \mathcal{G}) \) defined in Section 2, carries a strict contact covariant decomposable hyperbolic plane \( P \) (abr. s.c.c.d. hyperbolic plane) if:
1° the distribution \( P \) is contact covariant decomposable;
2° the dual forms of the null vectors which define \( P \) form an exterior recurrent pairing (in the sense of Rosca [7]).

Without loss of generality, one may suppose that \( P \) is defined by \( h_n \) and \( h_n^* = h_{2n} \).

In the first place, using (2.10) and (3.1), one finds
\[ n X_n = \sum_{i=1}^{n} \omega_i X_{2n} \] (3.3)
where \( \omega_n, \omega_{2n} \in \Lambda^1(\mathcal{M}) \) and \( \alpha \in \{i, i^*; i = 1, \ldots, n; i^* = i+n\} \).

Denote by \( P^\perp \) the complementary orthogonal distribution of \( P \) in \( D^n \).

Obviously one has \( P^\perp = \{h_a\} \), and we set
\[ X_n = x_n h_a \in P^\perp \] (3.4)
\[ X_{2n} = x_{2n} h_a \in P^\perp \]

Secondly, according to Rosca [7]; Morvan and Rosca [8], the dual forms \( \omega_n, \omega_{2n} \) corresponding to \( P = (h_n, h_{2n}) \) define an exterior recurrent pairing if one has
\[ d\omega_n = \gamma_n \Lambda \omega_n + \gamma_{2n} \Lambda \omega_{2n} \] (3.5)
\[ d\omega_{2n} = \gamma_{2n} \Lambda \omega_{2n} + \gamma_{2n} \Lambda \omega_{2n} \] (3.6)
\[ \gamma_n, \gamma, \gamma_{2n} \in \Lambda^1(\mathcal{M}) \] (3.7)

As a consequence of (3.5), using (2.12), (2.11), (2.14), and (3.3), we find:
\[ \gamma_n = \gamma_n (\frac{1}{n} x_{2n} \omega_{2n}), \] (3.8)
\[ \gamma_{2n} = \gamma_{2n} (\frac{1}{2n} x_{2n} \omega_{2n}) \]
where \( \gamma_n, \gamma_{2n} \in \Lambda^1(\mathcal{M}) \) vanish nowhere on \( M \). Therefore (3.5) become of the form
\[ d\omega_n = \gamma_n \Lambda \omega_n + \gamma_{2n} \Lambda \omega_{2n} \] (3.9)
\[ d\omega_{2n} = \gamma_{2n} \Lambda \omega_{2n} + \gamma_{2n} \Lambda \omega_{2n} \]
where we have set
\[ \gamma_n = \gamma_{2n} \] (3.10)
\[ \gamma = \gamma_n \Lambda \omega_{2n} \]

Denote now by
\[ \gamma = \gamma_n \Lambda \omega_{2n} \]
the simple unit form which corresponds to \( P \). It follows from (3.7) that
\[ d\gamma = 0. \]
Since \( \dim(\ker \gamma) \neq 0 \), we may also say that \( \gamma \) is a pre-symplectic form (Souriau [14]).
Further taking the exterior derivative of equations (3.7) and referring to (2.4), one gets by an easy argument that

\[ d\gamma = \hat{\gamma} + \gamma = \hat{\gamma}_n, \quad \gamma \in C^\infty(M). \]  

(3.11)

It follows from (3.11) that

\[ d\gamma = (d\hat{\gamma}/\hat{\gamma}) \wedge \gamma, \]  

(3.12)

i.e. \( \gamma \) is exterior recurrent and has the exact form \( d\hat{\gamma}/\hat{\gamma} \) as the recurrence 1-form.

Denote now by \( I(P^1) = \{ \omega \in \Lambda^1(M) : \omega \text{ annihilates } P^1 \} \) the ideal in \( \Lambda^*(M) \) of the distribution \( P^1 \). Obviously \( \hat{\gamma} \) belongs to this ideal and by means of (3.10) we may say that \( I(P^1) \) is a differentiable ideal \( (dI(P^1) \subseteq I(P^1)) \).

It follows as is known, that the distribution \( P^1 \) is involutive (this can be also checked by a direct computation with the help of (3.3) and (3.6)).

Let us now denote

\[ \hat{\gamma} = \omega^1 \wedge \ldots \wedge \omega^{n-1} \]  

(3.13)

the simple unit form corresponding to the distribution \( P^1 \). Then by means of (2.12), (2.11), (2.14), (3.3), (3.4) and (3.6), a straightforward calculation gives

\[ d\hat{\gamma} = (f_{n}^\gamma(X_n, X_n) \omega + f_{2n}^\gamma(X_{2n}, X_{2n}) \omega^{2n}) \wedge \hat{\gamma}. \]  

(3.14)

Hence the 2(n-1) form \( \hat{\gamma} \) is exterior recurrent and has the form

\[ \hat{\gamma} = f_{n}^\gamma(X_n, X_n) \omega + f_{2n}^\gamma(X_{2n}, X_{2n}) \omega^{2n} \]  

(3.15)

as a recurrence form (Datta [15]).

In the following we will call the vector field

\[ \tilde{\gamma} = f_{n}^\gamma(X_n, X_n) h_n + f_{2n}^\gamma(X_{2n}, X_{2n}) h_{2n} \]  

(3.16)

the recurrence vector field on \( M \) (\( g(\tilde{\gamma}, \tilde{\gamma}) = g(V, V) \)) and \( X_n, X_{2n} \) the distinguished vectors (abr. d.v.) of the distribution \( P^1 \).

By means of (2.6) one has

\[ \tilde{\gamma} \cdot \tilde{\gamma} = f_{2n}^\gamma(X_{2n}, X_{2n}) h_n - f_{n}^\gamma(X_n, X_n) h_{2n} \]  

(3.17)

and according to (2.8) this implies

\[ \tilde{d} \tilde{\gamma} = 0. \]  

(3.18)

Since \( \tilde{\gamma} \cdot \tilde{\gamma} \in P \), we have from (3.13), (3.14), and (3.18)

\[ \frac{1}{\tilde{\gamma} \cdot \tilde{\gamma}} \tilde{d} \tilde{\gamma} = 0, \]  

(3.19)

and the above equations proved that \( \tilde{\gamma} \) is a characteristic vector field of \( \hat{\gamma} \).

Moreover, if \( \tilde{X} \in P \) is any vector field of \( P \), one gets instantly \( \tilde{d} \tilde{X} = \tilde{\alpha}(\tilde{X}) \hat{\gamma} \), i.e. \( \tilde{X} \) is an infinitesimal conformal transformation of \( \hat{\gamma} \). Next the Ricci 2-form corresponding to \( P \) is \( \tilde{\alpha} = \tilde{\alpha}_n (\neq 2n) \), and it can be found by means of (2.14), (3.3) and (3.12):

\[ \frac{d\hat{\gamma}}{\hat{\gamma}} \wedge \hat{\gamma} = \tilde{\alpha}_n + \hat{\gamma} + \hat{\gamma}(X_n, X_{2n}) \hat{\gamma}_n \wedge \hat{\gamma}. \]  

(3.20)
Hence equations (3.12) and (3.10) show that the necessary and sufficient condition for $\Theta_n^2$ to be closed is that the vector fields $\tilde{X}_n$ and $\tilde{X}_{2n}$ are orthogonal.

Using now (3.11) and (3.9), one gets

$$\Theta_n^2(\tilde{X}_n, \tilde{X}_{2n}) = g(\tilde{X}_n, \tilde{X}_{2n})\langle X_n, X_{2n} \rangle + \Lambda_n^2 - \Lambda_{2n}^2 > 0.$$  \hspace{1cm} (3.21)

Therefore, if $\tilde{X}_n$ and $\tilde{X}_{2n}$ are orthogonal, then $\Theta_n^2(\tilde{X}_n, \tilde{X}_{2n})$ vanishes.

Denote now by $M^h$ the maximal connected integral manifold of $\mathbb{P}^h$ and let $H$ be the mean curvature $(2n-3)$-form of $M^h$. Then $H$ is defined by

$$H = \sum (-1)^{i-1} \omega \wedge \cdots \wedge \omega \wedge \omega^{n-1} \omega \wedge \cdots \wedge \omega^{n-1} \wedge h_i^*$$

(the roofs indicate the missing terms and we denote the induced elements on $M^h$ by supressing $\sim$ ). Since $\phi$ is the volume element of $M^h$, one has (see Chen [9])

$$dV^h = 2(n-1)\phi \wedge H$$  \hspace{1cm} (3.23)

where $H$ is the mean curvature vector field of $M^h$, $\phi = \frac{1}{H^4}$, and $dV^h$ is the exterior covariant differentiation with respect to $V^h$. Using (2.10), (2.12) and taking into account (2.14), (3.3), (3.6), and (3.16), one finds after some calculations

$$H = \frac{1}{2(n-1)} V; \hspace{1cm} V = \frac{1}{V^h}.$$  \hspace{1cm} (3.24)

Hence the mean curvature vector is, up to the factor $\frac{1}{2(n-1)}$, equal to the induced value of the recurrence vector field $\tilde{V}$ in $\tilde{M}$. Using the definition given by Rosca [16], [17], we obtain the following results:

1°. The necessary and sufficient condition for $M^h$ to be quasi-minimal

i.e., $H$ be a null vector field, is that one of the d.v. of the distribution $\mathbb{P}$ be a null vector.

2°. The necessary and sufficient condition for $M^h$ to be minimal is that both d.v. of $\mathbb{P}$ be null vectors.

We shall now make the following consideration. According to (2.21), (3.9) and (3.13) the volume element of the hypersurface $M_\eta$ defined by $\eta = 0$ may be written as:

$$\sigma = \phi \wedge \psi.$$  \hspace{1cm} (3.25)

In (3.25) $\phi$ and $\psi$ are the restrictions of $\tilde{\phi}$ and $\tilde{\psi}$ on $M_\eta$.

It follows from (3.10) that if one has $g(X_n, X_{2n}) = g(X_n, X_{2n}) = 0$, one may write $\delta \phi = 0$ where $\Lambda = d\phi + \delta \phi$ is the harmonic operator. Therefore we are in the situation of Tashibana's theorem (Tashibana [13]) and $M$ is covered by two families of minimal submanifolds, $M^h$ and $M$, tangent to $\mathbb{P}^h$ and $\mathbb{P}$ respectively.

Equations (2.6) shows that $\mathbb{P}^h = \mathbb{P}^0$ and $\mathbb{P} = \mathbb{P}$. Hence we may say that if both d.v. $X_n$ and $X_{2n}$ are null vectors, then $M_\eta$ is foliated by two families of invariant submanifolds tangent to $\mathbb{P}^h$ and $\mathbb{P}$, and therefore the whole manifold $\tilde{M}$ is foliate. Moreover, if we consider the immersion of $M$ in $M_\eta$, then the 1-forms $\theta_n^a, \theta_{2n}^a$ given by (3.3) define the normal vector quadratic form $II$ (it is known that $II$ is independent of the normal connection). But by means of (3.6) we can see...
that II vanishes, and therefore $M$ is totally geodesic in $M_n$.

We shall give now the following

DEFINITION. Let $M$ be an invariant submanifold of a manifold $\tilde{M}$ endowed with a para $f$-structure and II be the normal vector quadratic form of $M$. Then any tangent vector field $X$ of $M$ such that $II(X,fX)=0$ is called an $f$-geodesic direction on $M$.

Let us consider now the immersion $x: M \rightarrow M$. Denote by $\varepsilon_n = \langle dp, Vh_n \rangle$ and $\varepsilon_{2n} = \langle dp, Vh_{2n} \rangle$ the second quadratic forms associated with $x$.

By means of (2.9), (2.10), (3.3), and (3.6) one finds after some calculation

$$\varepsilon_n = \frac{1}{f_n} \pi_n \otimes \pi_n = f_n \left( \sum_\alpha x_n^\alpha \omega^\alpha \right)^2,$$

$$\varepsilon_{2n} = \frac{1}{f_{2n}} \pi_{2n} \otimes \pi_{2n} = f_{2n} \left( \sum_\alpha x_{2n}^\alpha \omega^\alpha \right)^2.$$

(3.26)

Therefore the normal vector quadratic form $II \in (T^*T^\perp) \otimes (T^*T^\perp)$ is given by

$$II = \frac{1}{f_n} (\pi \otimes \pi) \otimes h_n + \frac{1}{f_{2n}} (\pi_{2n} \otimes \pi_{2n}) \otimes h_{2n}.$$

(3.27)

Referring now to (2.4), one gets by means of (2.26) and (2.27)

$$II(X_n, UX_n) = 0, \quad II(X_{2n}, UX_{2n}) = 0.$$

Therefore the d.v. fields on $\mathcal{M}$ are both $U$-geodesic.

THEOREM 2. Let $M(U, \tilde{M}, \tilde{f}, \tilde{g})$ be an a.c.p. $f$-manifold admitting a strict contact covariant decomposable hyperbolic plane $P$ and $P^\perp$ be the orthogonal component of $P$ in the horizontal distribution $D_n$. Further let $V \in P$ and $X_n, X_{2n} \in P^\perp$ be the recurrence vector field and the distinguished vector fields associated with the pairing $(P, P^\perp)$.

Then the following properties hold:

(i) The distribution $P^\perp$ is always involutive and the mean curvature vector field of the maximal integral manifold $\mathcal{M}$ of $P^\perp$ is (up to a constant factor) equal to the induced vector field of $\tilde{V}$.

(ii) The simple unit form $\tilde{\psi}$ of $P^\perp$ is exterior recurrent and $UV$ is a characteristic vector field of $\tilde{\psi}$.

(iii) The necessary and sufficient condition for $\mathcal{M}$ to be quasi-minimal is that one of the d.v. fields of $\mathcal{M}$ be a null vector and the necessary and sufficient condition for $\mathcal{M}$ to be minimal is that both d.v. fields of $\mathcal{M}$ be null vectors.

(iv) If $\mathcal{M}$ is minimal, then the distribution $P$ is also involutive and the integral surfaces of $P$ are totally geodesic in $M_n$, which in this case is foliate.

(v) Both d.v. fields on $\tilde{\mathcal{M}}$ are $U$-geodesic directions on $\mathcal{M}$.

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