RESEARCH NOTES

A RELATIONSHIP BETWEEN THE MODIFIED EULER METHOD AND e

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ABSTRACT. Approximating solutions to the differential equation $dy/dx = f(x,y)$ where $f(x,y) = y$ by a generalization of the modified Euler method yields a sequence of approximates that converge to e . Bounds on the rapidity of convergence are determined, with the fastest convergence occurring when the parameter value is $\frac{1}{2}$, so the generalized method reduces to the standard modified Euler method. The situation is similarly examined when f is altered.

KEY WORDS AND PHRASES. *Euler method, modified Euler method.*

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1. INTRODUCTION.

The Euler method is known as a simple, but crude, method for approximating solutions to differential equations. The modified Euler method offers greater refinement, as shown in Ross [1]. Let us recall in this setting we wish to solve the equation $dy/dx = f(x,y)$ subject to the condition $y(x_0) = y_0$. We let h denote a positive increment in x and define $x_k = x_0 + kh$. To approximate the exact solution y at x_k , $y(x_k) = y_k$, we construct a sequence of approximates $y_k^{(1)}, y_k^{(2)}, \dots$ which converge to y_k . Proceeding inductively we get y_{k+1} by considering the sequence:

$$y_{k+1}^{(1)} = y_k + hf(x_k, y_k) \quad (1.1)$$

$$y_{k+1}^{(2)} = y_k + (h/2)[f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(1)})] \quad (1.2)$$

while in general

$$y_{k+1}^{(n)} = y_k + (h/2)[f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(n-1)})]. \quad (1.3)$$

When successive terms in this sequence are close enough, we set their common value equal to y_{k+1} . With this in mind, we can consider the equation

$$y_{k+1} = y_k + h[\frac{1}{2}f(x_k, y_k) + \frac{1}{2}f(x_{k+1}, y_{k+1})] \quad (1.4)$$

as defining the solution points by the modified Euler method (MEM).

If we consider the specific differential equation with $f(x, y) = y$, and the side condition $y(0) = 1$, with an increment of $h = 1/n$ we get the values

$$y_n = \left[\frac{2n+1}{2n-1} \right]^n = \left(1 + \frac{1}{n-.5} \right)^{(n-.5) + .5} \quad (1.5)$$

This produces a sequence that converges to e (as was to be expected since $y' = y$).

The modified Euler method fits into a more general scheme given by

$$y_{k+1} = y_k + h[pf(x_k, y_k) + (1-p)f(x_{k+1}, y_{k+1})] \quad (1.6)$$

where $0 \leq p \leq 1$. If we now apply this generalized method (call it M_p EM) to the same differential equation as above, we get a general term of

$$y_n = \left[\frac{n+p}{n-(1-p)} \right]^n = \left[1 + \frac{1}{n-(1-p)} \right]^n \quad (1.7)$$

and clearly y_n approaches e . We note here that $p = 1$ produces the same sequence as the Euler method, and $p = \frac{1}{2}$ produces the same sequence as the modified Euler method.

2. MAIN RESULTS.

One is now led to ask which value of p yields the sequence that best approximates e . The expression for y_n suggests one could examine the family of functions

$$f_p(x) = (1 + 1/x)^{x + (1-p)}. \quad (2.1)$$

These functions fall into one of three types, depending on the size of p . The function f_p is decreasing for $p \leq .5$, is increasing for $p > (-1 + \sqrt{5})/2$, and is decreasing at first then eventually increasing for $.5 < p < (-1 + \sqrt{5})/2$. The reader is referred to the articles by Darst, Dence and Polya [2-4] for further details on this. It follows that $p = .5$ yields the best approximation to e because any value p' greater than $.5$ can be improved upon by, say, $(p' + .5)/2$. Perhaps Euler knew something that we haven't given him credit for when he chose $p = \frac{1}{2}$ instead of an alternate weighting system!

To determine how quickly $f_{.5}$ converges to e , we wish to find N such that $x > N$ implies $|f_{.5}(x) - e|$ is bounded above by $\epsilon > 0$. To this end we have

$$f_{.5}(x) - f_{.51}(x) = (1 + \frac{1}{x})^x [e^{.501 \ln(1 + 1/x)} - e^{.491 \ln(1 + 1/x)}] \quad (2.2)$$

$$= (1 + \frac{1}{x})^x \sum_{i=0}^{\infty} (.50^i - .49^i) \ln^i(1 + 1/x) (1/i!) \quad (2.3)$$

$$< e \sum_{i=0}^{\infty} .01(i) (.50^{i-1}) \ln^i(1 + 1/x) (1/i!) \quad (2.4)$$

$$< .01e \sum_{i=1}^{\infty} i (.50^{i-1}) (1/x)^i (1/i!) \quad (2.5)$$

$$< .01e \sum_{i=1}^{\infty} 2^{1-i} x^{-i} \quad (2.6)$$

$$= e/[50(2x - 1)], \quad \text{for } |2x| > 1. \quad (2.7)$$

Since $f_{.5}(x) - e < .5[f_{.5}(x) - f_{.51}(x)] < e/[100(2x - 1)]$ for all sufficiently large x , the difference between $f_{.5}$ and e can be made small enough by choosing x greater than $.5[1 + e/(100e)]$. For example, with $\epsilon = .0001$ we then choose $x > 136.4$ and get $f_{.5}(137) = 2.7182938$ and the difference $f_{.5}(137) - e = .000012$.

If we now consider the slightly more general initial value problem $dy/dx = f(x, y) = Ay$ with side condition $y(0) = 1$ then, using M_{pEM} , we get

$$y_1 = y_0 + (1/n)[pAy_0 + (1-p)Ay_1] = 1 + (1/n)[pA + (1-p)Ay_1] \quad (2.8)$$

so

$$y_1 = \frac{n + Ap}{n + (p-1)A} \quad (2.9)$$

and then

$$y_2 = y_1 + (1/n)[pAy_1 + (1-p)Ay_2] \quad (2.10)$$

so $y_2 = y_1^2$. The n -th term is given by $y_n = y_1^n$, or

$$y_n = \left[1 + \frac{A}{n - (1-p)A}\right]^n \quad (2.11)$$

so y_n converges to e^A . Furthermore, since y_n is of the form $(1 + A/x)^x + (1-p)A$, insight into the behavior of y_n can be gained by examining the related family of sequences

$$(1 + A/n)^{Bn + C} \quad (2.12)$$

with A, B, C real. We shall consider A as positive in what follows.

Case 1. Set $a_n = (1 + A/n)^{n + \alpha}$ and $b_n = (1 + A/n)^{-n + \alpha}$ and define the number $\gamma(A)$ by

$$\gamma(A) = \frac{2 \ln(1 + A/2) - \ln(1 + A)}{\ln(1 + A) - \ln(1 + A/2)} > 0. \quad (2.13)$$

The motivation for this is that $\gamma(A)$ is the limiting value of α as n tends to ∞ for which $a_n = a_{n+1}$. By methods analagous to those used by the author in [3] we know that $\{a_n\}$ is increasing if $\alpha < \gamma(A)$, decreasing if $\alpha \geq A/2$, and initially decreasing then eventually increasing if $\gamma(A) < \alpha < A/2$. Because b_n is basically a reciprocal of a_n it follows that the monotonicity of $\{b_n\}$ is increasing if $\alpha \leq -A/2$, decreasing if $\alpha > -\gamma(A)$, and initially increasing then eventually decreasing if $-A/2 < \alpha < -\gamma(A)$. Case 2. Set $c_n = (1 - A/n)^{n + \alpha}$ and $d_n = (1 - A/n)^{-n + \alpha}$, with $n > A$, and define the number $\gamma(A)$ by

$$\gamma(A) = \frac{([A] + 2)\ln(1 - \frac{A}{[A] + 2}) - ([A] + 1)\ln(1 - \frac{A}{[A] + 1})}{\ln(1 - \frac{A}{[A] + 1}) - \ln(1 - \frac{A}{[A] + 2})} < 0 \quad (2.14)$$

where the brackets denote the greatest integer function. Similar to above we have that $\{c_n\}$ is increasing if $\alpha \geq -A/2$, decreasing if $\alpha < \gamma(A)$, and initially increasing then eventually decreasing if $\gamma(A) < \alpha < -A/2$, and that $\{d_n\}$ is increasing if $\alpha > -\gamma(A)$, decreasing if $\alpha \leq A/2$, and initially decreasing then eventually increasing if

$A/2 < \alpha < -\gamma(A)$. Because of cases 1 and 2 we can determine the monotonicity of (2.12) from the identity

$$\left(1 + \frac{A}{n}\right)^{Bn + C} = \left[\left(1 + \frac{A}{n}\right)^{(\text{sgn } B)n + C/|B|}\right]^{|B|}. \quad (2.15)$$

Furthermore, since (2.11) is of the form

$$\left(1 + \frac{A}{x}\right)^x + (1-p)A \quad (2.16)$$

it follows that the fastest convergence to e^A is when $(1-p)A = A/2$, or $p = \frac{1}{2}$. This is because $\left(1 + \frac{A}{n}\right)^{n + \alpha}$ is decreasing to e^A for $\alpha \geq A/2$, with the fastest convergence at $\alpha = A/2$. We remark here that some of the above monotonicity properties could be alternately derived by examining the logarithm of $\left(1 + \frac{A}{x}\right)^{Bx + C}$.

The rapidity of this convergence can be discussed by considering the functions $f_p(x)$, given by (2.16), and noting (same technique as before) that,

$$f_{.5}(x) - f_{.51}(x) < e^A [e^{.50A \ln(1 + A/x)} - e^{.49A \ln(1 + A/x)}] \quad (2.17)$$

$$< .01e^A \sum_{i=1}^{\infty} A^i (A/x)^i 2^{1-i} \quad (2.18)$$

$$= A^2 e^A / [50(2x - A^2)]. \quad (2.19)$$

Table 1 lists some data for this situation.

x	$f_{.50}(x)$	$f_{.51}(x)$	$f_{.50}(x) - f_{.51}(x)$	$\frac{A^2 e^A}{50(2x - A^2)}$
10	20.43377	20.27357	.16020	.32867
50	20.10259	20.06748	.03511	.03972
100	20.08992	20.07211	.01781	.01892
400	20.08581	20.08131	.00450	.00457

Table 1 ($A = 3$)

For large enough x we have

$$f_{.50}(x) - e^A < \frac{1}{2}[f_{.50}(x) - f_{.51}(x)] < \frac{A^2 e^A}{100(2x - A^2)} \quad (2.20)$$

and for this difference to be less than $\epsilon > 0$ just choose x greater than $.5[A^2 + A^2 e^A / (100\epsilon)]$. For example, with $\epsilon = .001$, we choose $x = 999$ and get $f_{.50}(x) - e^3 = .00004$.

3. CONCLUDING REMARKS.

Noticing how critical the value $p = \frac{1}{2}$ is on the efficiency of convergence prompts one to characterize those functions $f(x, y)$ which fall under this classification. Knowing this to be true for $f(x, y) = Ay$, we can now show it to be true for the elementary functions $f(x, y) = x^m$, with the side condition $(0, 0)$, and for $m = 0, 1, 2, 3, \dots$: (we know $y = x^{m+1}/(m+1)$ and $y(1) = 1/(m+1)$). Using M_p^{EM} of (1.6) and $\gamma = 1/n$ we get

$$y_n = \frac{(1-p) \sum_{i=1}^n i^m + p \sum_{i=1}^{n-1} i^m}{n^{m+1}} = \frac{\sum_{i=1}^n i^m - pn^m}{n^{m+1}}. \quad (3.1)$$

But $\sum_{i=1}^n i^m$ is expressible as a polynomial $p(n) = \sum_{i=1}^{m+1} a_i n^i$ with $a_{m+1} = 1/(m+1)$ and

$a_m = \frac{1}{2}$. Thus y_n can be written as

$$y_n = \frac{\left(\frac{1}{m+1} n^{m+1} + a_{m-1} n^{m-1} + \dots + a_1 n \right) + \left(\frac{1}{2} n^m - pn^m \right)}{n^{m+1}} \quad (3.2)$$

and this expression converges to $1/(m+1)$ fastest when $p = \frac{1}{2}$. Likewise it follows that $p = \frac{1}{2}$ whenever $f(x,y)$ is a polynomial in x . Further classifications of f appear to be more difficult to obtain.

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