ON A FUNCTION RELATED TO RAMANUJAN'S TAU FUNCTION

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ABSTRACT. For the function \( \psi = \psi_{12} \), defined by \( \sum_{1}^{m} \psi(n)x^n = x \prod_{1}^{m} (1-x^{2n})^{12} \quad (|x|<1) \), the author derives two simple formulas. The simpler of these two formulas is expressed solely in terms of the well-known sum-of-divisors function.

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1. INTRODUCTION.

Following Ramanujan [4,p. 155] we define for each positive divisor \( \alpha \) of 24 an arithmetical function \( \psi_{\alpha} \) as follows:

\[
\psi_{\alpha}(n)x^n = x \prod_{1}^{m} (1-x^{2n/\alpha})^{\alpha},
\]

an identity which is valid for each complex number \( x \) such that \( |x| < 1 \). Of course, \( \psi_{24} = \tau \), the celebrated Ramanujan tau function. In this paper we are specifically concerned with \( \psi_{12} \) (\( \psi \) for simplicity). As a matter of fact, we derive two explicit formulas for \( \psi \).

Since these formulas involve the sum-of-divisors function and the counting function for sums of eight squares, we need the following definition.

Definition. (i) For each positive integer \( n \), \( \sigma(n) \) denotes the sum of all positive divisors of \( n \). (ii) For each nonnegative integer \( n \), \( r_k(n) \) denotes the cardinality of the set

\[
\{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k \mid n = x_1^2 + x_2^2 + \ldots + x_k^2\},
\]

\( k \) an arbitrary positive integer.

We can now state our main result.

Theorem 1. For each nonnegative integer \( m \),

\[
\psi(2m+1) = \sum_{i=0}^{m} (-1)^i r_8(i) \sigma(2m-2i+1), \quad (1.2)
\]

\[
\psi(2m+2) = 0. \quad (1.3)
\]

In section 2 we prove theorem 1, and thereafter prove a corollary which gives a formula expressing \( \psi \) solely in terms of \( \sigma \).

2. PROOF OF THEOREM 1. Our proof requires the following three identities, each of which is valid for each complex number \( x \) such that \( |x| < 1 \).

\[
\prod_{1}^{m} (1+x^{2n-1}) = 1 \quad (2.1)
\]
Identity (2.1) is due to Euler, while (2.2) and (2.3) are due to Gauss. For proofs see [3, pp. 277-284]. We also need a fourth identity which the author has not been able to locate in the literature. This we here record in the following lemma.

**LEMMA.** For each complex number $x$ such that $|x| < 1$,

$$\sum_{m=0}^{\infty} x^m(m+1)/2 = \frac{\pi}{2} \cot \pi x$$

(2.4)

**Proof:** Here we need the following two identities, stated and proved in [1, p. 313].

$$\prod_{l} (1-x^{2n})^2 (1+x^{2n-1})^2 = \left( \sum_{m=0}^{\infty} x^{2m} \right)^2 + x \left( \sum_{m=0}^{\infty} x^{2m(m+1)} \right)^2$$

$$(1-x^{2n})^2 (1-x^{2n-1})^2 = \left( \sum_{m=0}^{\infty} x^{2m} \right)^2 - x \left( \sum_{m=0}^{\infty} x^{2m(m+1)} \right)^2$$

We square these identities, add the resulting identities, and utilize the fact that the fourth power of the right side of (2.2) generates $(-1)^{n+1} r_4(n)$, to write:

$$\sum_{l=0}^{\infty} r_4(2n)x^{2n} = \sum_{n=0}^{\infty} r_4(n)x^n + \left( \sum_{n=0}^{\infty} (-1)^{n+1} r_4(n)x^n \right)^4$$

$$= 2 \sum_{n=0}^{\infty} r_4(n)x^{2n} + 2x^2 \sum_{n=0}^{\infty} x^{2m(m+1)}$$

whence

$$\sum_{n=0}^{\infty} x^{2m(m+1)} = \sum_{m=0}^{\infty} r_4(2n)x^{2n}$$

We then raise the identity to the fourth power, and multiply the resulting identity by the eighth power of identity (2.2) to get

$$\prod_{l} (1-x^n)(1-x^{2n-1})^{-2} = \sum_{n=0}^{\infty} x^n(n+1)/2$$

Continuing with the proof of theorem 1, we use (2.1) to rewrite (2.3) as

We then raise the identity to the fourth power, and multiply the resulting identity by the eighth power of identity (2.2) to get

$$\prod_{l} (1-x^n)^{12} = \sum_{n=0}^{\infty} (-x)^{n+1} \sum_{j=0}^{\infty} \sigma(2j+1)x^j$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^i r_8(i) \sigma(2n-2i+1)$$
In the foregoing we then let \( x \mapsto x^2 \), and multiply the resulting identity by \( x \) to get

\[
\sum_{n=1}^{\infty} \psi(n)x^n = x\sum_{n=1}^{\infty} (1-x^{2n})^{1/2}
\]

\[
= \sum_{n=0}^{\infty} x^{2m+1} \sum_{i=0}^{m} (-1)^{i} r_8(i) \sigma(2m-2i+1)
\]

Comparing coefficients of \( x^n \) we thus prove our theorem.

By appeal to the well-known formula for \( r_8 \), viz.,

\[
r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3, \quad n \in \mathbb{Z}^+
\]

(e.g., see [3, p. 314]), we eliminate \( r_8 \) from (1.2) as follows:

\[
\psi(2m+1) = \sigma(2m+1) + 16 \sum_{i=1}^{m} \sigma(2m-2i+1) \sum_{d|1} (-1)^d d^3
\]

In order to extend the inner sum over all \( d \) in the range \( 1,2,...,i \) we define \( \varepsilon(i,d) \) to be 1, if \( d \) divides \( i \), to be 0, otherwise. Hence,

\[
\psi(2m+1) = \sigma(2m+1) + 16 \sum_{i=1}^{m} \sum_{d=1}^{i} (-1)^d \sigma(2m-2i+1) \varepsilon(i,d) d^3
\]

\[
= \sigma(2m+1) + 16 \sum_{d=1}^{m} (-1)^d d^3 \sum_{i=d}^{m} \varepsilon(i,d) \sigma(2m-2i+1)
\]

\[
= \sigma(2m+1) + 16 \sum_{d=1}^{m} (-1)^d d^3 \sum_{k=1}^{\lfloor m/d \rfloor} \sigma(2m-2kd+1)
\]

The upper limit of summation of the sum indexed by \( k \) is naturally \( \lfloor m/d \rfloor \), the integral part of \( m/d \). Thus, we have proved the following

**COROLLARY.** For each nonnegative integer \( m \),

\[
\psi(2m+1) = \sigma(2m+1) + 16 \sum_{d=1}^{\lfloor m/d \rfloor} (-1)^d d^3 \sigma(2m-2kd+1)
\]

**CONCLUDING REMARKS.** According to Hardy, Ramanujan conjectured that each of the \( \psi_a \) (for a dividing 24) is multiplicative; e.g., see [2, p. 184]. These conjectures were later confirmed by L. J. Mordell. Owing to classical identities of Euler and Jacobi, \( \psi_1 \) and \( \psi_3 \) are trivially defined. Ramanujan himself deduced formulas for \( \psi_2 \), \( \psi_4 \), \( \psi_6 \) and \( \psi_8 \).

**REFERENCES**

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