ON A FUNCTION RELATED TO RAMANUJAN'S TAU FUNCTION

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ABSTRACT. For the function $\psi = \psi_{12}$, defined by $\sum_{n=1}^{\infty} \psi(n)x^n = x \prod_{n=1}^{\infty} (1-x^{2n})^{12}$ ($|x| < 1$), the author derives two simple formulas. The simpler of these two formulas is expressed solely in terms of the well-known sum-of-divisors function.

KEY WORDS AND PHRASES. Ramanujan's tau function, related arithmetical functions.

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1. INTRODUCTION.

Following Ramanujan [4, p. 155] we define for each positive divisor α of 24 an arithmetical function ψ_{α} as follows:

$$\sum_{n=1}^{\infty} \psi_{\alpha}(n)x^n = x \prod_{n=1}^{\infty} (1-x^{24n/\alpha})^{\alpha}, \quad (1.1)$$

an identity which is valid for each complex number x such that $|x| < 1$. Of course, $\psi_{24} = \tau$, the celebrated Ramanujan tau function. In this paper we are specifically concerned with ψ_{12} ($=\psi$ for simplicity). As a matter of fact, we derive two explicit formulas for ψ . Since these formulas involve the sum-of-divisors function and the counting function for sums of eight squares, we need the following definition.

Definition. (i) For each positive integer n , $\sigma(n)$ denotes the sum of all positive divisors of n . (ii) for each nonnegative integer n , $r_k(n)$ denotes the cardinality of the set

$$\{(x_1, x_2, \dots, x_k) \in \mathbb{Z}^k \mid n = x_1^2 + x_2^2 + \dots + x_k^2\},$$

k an arbitrary positive integer.

We can now state our main result.

Theorem 1. For each nonnegative integer m ,

$$\psi(2m+1) = \sum_{i=0}^m (-1)^i r_8(i) \sigma(2m-2i+1), \quad (1.2)$$

$$\psi(2m+2) = 0. \quad (1.3)$$

In section 2 we prove theorem 1, and thereafter prove a corollary which gives a formula expressing ψ solely in terms of σ .

2. PROOF OF THEOREM 1. Our proof requires the following three identities, each of which is valid for each complex number x such that $|x| < 1$.

$$\prod_{n=1}^{\infty} (1+x^n)(1-x^{2n-1}) = 1 \quad (2.1)$$

$$\prod_{l=1}^{\infty} (1-x^n)(1-x^{2n-1}) = \sum_{n=-\infty}^{\infty} (-x)^n n^2 \quad (2.2)$$

$$\prod_{l=1}^{\infty} (1-x^{2n})(1+x^n) = \sum_{n=0}^{\infty} x^{n(n+1)/2} \quad (2.3)$$

Identity (2.1) is due to Euler, while (2.2) and (2.3) are due to Gauss. For proofs see [3, pp. 277-284]. We also need a fourth identity which the author has not been able to locate in the literature. This we here record in the following lemma.

LEMMA. For each complex number x such that $|x| < 1$,

$$\left\{ \sum_{n=0}^{\infty} x^{n(n+1)/2} \right\}^4 = \sum_{n=0}^{\infty} \sigma(2n+1) x^n \quad (2.4)$$

Proof: Here we need the following two identities, stated and proved in [1, p. 313].

$$\prod_{l=1}^{\infty} (1-x^{2n})^2 (1+x^{2n-1})^4 = \left\{ \sum_{n=-\infty}^{\infty} x^{2n^2} \right\}^2 + x \left\{ \sum_{n=-\infty}^{\infty} x^{2n(m+1)} \right\}^2$$

$$\prod_{l=1}^{\infty} (1-x^{2n})^2 (1-x^{2n-1})^4 = \left\{ \sum_{n=-\infty}^{\infty} x^{2n^2} \right\}^2 - x \left\{ \sum_{n=-\infty}^{\infty} x^{2n(m+1)} \right\}^2$$

We square these identities, add the resulting identities, and utilize the fact that the fourth power of the right side of (2.2) generates $(-1)^n r_4(n)$, to write:

$$\begin{aligned} 2 \sum_{n=0}^{\infty} r_4(2n) x^{2n} &= \sum_{n=0}^{\infty} r_4(n) x^n + \sum_{n=0}^{\infty} (-1)^n r_4(n) x^n \\ &= 2 \sum_{n=0}^{\infty} r_4(n) x^{2n} + 2x^2 \left\{ \sum_{n=-\infty}^{\infty} x^{2n(m+1)} \right\}^4, \end{aligned}$$

whence

$$\begin{aligned} x^2 \left\{ \sum_{n=-\infty}^{\infty} x^{2n(m+1)} \right\}^4 &= \sum_{n=0}^{\infty} [r_4(2n) - r_4(n)] x^{2n} \\ &= \sum_{n=0}^{\infty} [r_4(4n) - r_4(2n)] x^{4n} \\ &\quad + \sum_{n=0}^{\infty} [r_4(4n+2) - r_4(2n+1)] x^{4n+2} \\ &= \sum_{n=0}^{\infty} [24\sigma(2n+1) - 8\sigma(2n+1)] x^{4n+2} \\ &= 2^4 \sum_{n=0}^{\infty} \sigma(2n+1) x^{4n+2} \end{aligned}$$

Here, we've made use of Jacobi's formula for $r_4(n)$. Now, cancelling $2^4 x^2$ and subsequently letting $x \rightarrow x^{1/4}$, we obtain (2.4).

Continuing with the proof of theorem 1, we use (2.1) to rewrite (2.3) as

$$\prod_{l=1}^{\infty} (1-x^n)(1-x^{2n-1})^{-2} = \sum_{n=0}^{\infty} x^{n(n+1)/2}$$

We then raise the identity to the fourth power, and multiply the resulting identity by the eighth power of identity (2.2) to get

$$\begin{aligned} \prod_{l=1}^{\infty} (1-x^n)^{12} &= \left\{ \sum_{n=-\infty}^{\infty} (-x)^n n^2 \right\}^8 \left\{ \sum_{n=0}^{\infty} x^{n(n+1)/2} \right\}^4 \\ &= \sum_{i=0}^{\infty} (-1)^i r_8(i) x^i \cdot \sum_{j=0}^{\infty} \sigma(2j+1) x^j \\ &= \sum_{n=0}^{\infty} x^n \sum_{i=0}^n (-1)^i r_8(i) \sigma(2n-2i+1). \end{aligned}$$

In the foregoing we then let $x \rightarrow x^2$, and multiply the resulting identity by x to get

$$\begin{aligned}\sum_{n=1}^{\infty} \psi(n)x^n &= x \cdot \prod_{n=1}^{\infty} (1-x^{2n})^{1/2} \\ &= \sum_{m=0}^{\infty} x^{2m+1} \sum_{i=0}^m (-1)^i r_8(i) \sigma(2m-2i+1)\end{aligned}$$

Comparing coefficients of x^n we thus prove our theorem.

By appeal to the well-known formula for r_8 , viz.,

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3, \quad n \in \mathbb{Z}^+$$

(e.g., see [3, p. 314]), we eliminate r_8 from (1.2) as follows:

$$\psi(2m+1) = \sigma(2m+1) + 16 \sum_{i=1}^m \sigma(2m-2i+1) \sum_{d|i} (-1)^d d^3$$

In order to extend the inner sum over all d in the range $1, 2, \dots, i$ we define $\epsilon(i, d)$ to be 1, if d divides i , to be 0, otherwise. Hence,

$$\begin{aligned}\psi(2m+1) &= \sigma(2m+1) + 16 \sum_{i=1}^m \sum_{d=1}^i (-1)^d \sigma(2m-2i+1) \epsilon(i, d) d^3 \\ &= \sigma(2m+1) + 16 \sum_{d=1}^m (-1)^d d^3 \sum_{i=d}^m \epsilon(i, d) \sigma(2m-2i+1) \\ &= \sigma(2m+1) + 16 \sum_{d=1}^m (-1)^d d^3 \sum_{k=1}^{m/d} \sigma(2m-2kd+1)\end{aligned}$$

The upper limit of summation of the sum indexed by k is naturally $[m/d]$, the integral part of m/d . Thus, we have proved the following

COROLLARY. For each nonnegative integer m ,

$$\psi(2m+1) = \sigma(2m+1) + 16 \sum_{d=1}^m (-1)^d d^3 \sum_{k=1}^{m/d} \sigma(2m-2kd+1).$$

CONCLUDING REMARKS. According to Hardy, Ramanujan conjectured that each of the ψ_α (for α dividing 24) is multiplicative; e.g., see [2, p. 184]. These conjectures were later confirmed by L. J. Mordell. Owing to classical identities of Euler and Jacobi, ψ_1 and ψ_3 are trivially defined. Ramanujan himself deduced formulas for ψ_2 , ψ_4 , ψ_6 and ψ_8 .

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