ON LOCALLY CONFORMAL KÄHLER SPACE FORMS

KOJI MATSUMOTO

Department of Mathematics
Faculty of Education
Yamagata University
Yamagata, 990, Japan

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ABSTRACT. An $m$-dimensional locally conformal Kähler manifold (l.c.K-manifold) is characterized as a Hermitian manifold admitting a global closed 1-form $\alpha_\lambda$ (called the Lee form) whose structure $(F^\lambda_\mu, g_{\mu\lambda})$ satisfies

\[ \nabla^*_\nu F^\mu_\lambda = -\beta^\mu_\nu g^\nu_\lambda + \beta^\lambda_\nu g^\mu_\nu = \alpha^\lambda_\mu F^\nu_\lambda + \alpha^\mu_\lambda F^\nu_\mu, \]

where $\nabla^*_\nu$ denotes the covariant differentiation with respect to the Hermitian metric $g_{\mu\lambda}$, $\beta^\mu_\nu = -F^\mu_\lambda g^\nu_\lambda$, $F^\mu_\lambda = F^\mu_\nu g^\nu_\lambda$ and the indices $\nu, \mu, \ldots, \lambda$ run over the range $1, 2, \ldots, m$.

For l.c.K-manifolds, I. Vaisman [4] gave a typical example and T. Kashiwada ([1], [2], [3]) gave a lot of interesting properties about such manifolds.

In this paper, we shall study certain properties of l.c.K-space forms. In §2, we shall mainly get the necessary and sufficient condition that an l.c.K-space form is an Einstein one and the Riemannian curvature tensor with respect to $g_{\mu\lambda}$ will be expressed without the tensor field $F_{\mu\lambda}$. In §3, we shall get the necessary and sufficient condition that the length of the Lee form is constant and the sufficient condition that a compact l.c.K-space form becomes a complex space form. In the last §4, we shall prove that there does not exist a non-trivial recurrent l.c.K-space form.


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1. INTRODUCTION.

This paper is directed to specialist readers with background in the area and appreciative of its relation of this area of study.

Let $M(F^\lambda_\mu, g_{\mu\lambda}, \alpha_\lambda)$ be an l.c.K-manifold. Then, by the definition, at any point of $M$ there exists a neighborhood in which a conformal metric $g^* = e^{-2\phi}g$ is a Kähler one, i.e.,

\[ \nabla^*_\nu (e^{-2\phi}F^\mu_\lambda) = 0, \quad d\phi = \alpha, \]

where $\nabla^*_\nu$ denotes the covariant differentiation with respect to $g^*$. Then we have
The following proposition was proved by T. Kashiwada [1]

PROPOSITION 1.1. A Hermitian manifold $M(F, g)$ is an l.c.K-manifold if and only if there exists a global closed 1-form $\alpha_\lambda$ satisfying (1.1).

In an l.c.K-manifold $M$, we define a tensor field $P_{\mu\lambda}$ as follows;

$$P_{\mu\lambda} = -\nabla_{\mu} \alpha_\lambda - \alpha_{\mu} \alpha_\lambda + \frac{1}{2} \|\alpha\|^2 g_{\mu\lambda},$$

where $\|\alpha\|$ denotes the length of the Lee form $\alpha_\lambda$ with respect to $g_{\mu\lambda}$.

In an m-dimensional l.c.K-manifold $M$, we know the following formula;

$$R_{\mu\nu} F_{\lambda\gamma} + R_{\lambda\nu} F_{\mu\gamma} - (m - 2)(P_{\mu\nu} F_{\lambda\gamma} + P_{\lambda\nu} F_{\mu\gamma}) = 0,$$

where $R_{\mu\lambda}$ denotes the Ricci tensor with respect to $g_{\mu\lambda}$ [1]. Thus we have

PROPOSITION 1.2. In an m-dimensional $(m \neq 2)$ l.c.K-manifold $M$, the tensor field $P_{\mu\lambda}$ is hybrid, i.e.,

$$P_{\mu\nu} F_{\lambda\gamma} + P_{\lambda\nu} F_{\mu\gamma} = 0,$$

if and only if the Ricci tensor $R_{\mu\lambda}$ is hybrid.

From now on in this paper, we assume that the tensor field $P_{\mu\lambda}$ is hybrid.

REMARK. In an m-dimensional $(m \neq 2)$ Einstein l.c.K-manifold, the tensor field $P_{\mu\lambda}$ is hybrid, identically.

An l.c.K-manifold $M$ is called an l.c.K-space form if the holomorphic sectional curvature of the section $\{X, FX\}$ at each point of $M$ has the constant value. Let $M(H)$ be an l.c.K-space form with constant holomorphic sectional curvature $H$. Then the Riemannian curvature tensor $R_{\omega\nu\mu\lambda}$ with respect to $g_{\mu\lambda}$ can be written as

$$\Delta R_{\omega\nu\mu\lambda} = H(g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda} + F_{\omega\lambda} F_{\nu\mu} - F_{\omega\mu} F_{\nu\lambda} - 2 F_{\omega\nu} F_{\mu\lambda}) + 3(P_{\omega\lambda} g_{\nu\mu} - P_{\omega\mu} g_{\nu\lambda}) + g_{\omega\lambda} P_{\nu\mu} - g_{\omega\mu} P_{\nu\lambda} - \{\overline{P}_{\omega\lambda} F_{\nu\mu} - \overline{P}_{\omega\mu} F_{\nu\lambda} + F_{\omega\nu} \overline{P}_{\lambda\mu} - F_{\omega\nu} \overline{P}_{\lambda\mu} - 2(\overline{P}_{\omega\nu} F_{\lambda\mu} + P_{\omega\nu} F_{\lambda\mu}),$$

where $\overline{P}_{\mu\lambda} = P_{\mu\lambda} F_{\nu\lambda}$ [1].

2. L.C.K-SPACE FORMS.

In this section, we shall consider the necessary and sufficient condition that an l.c.K-space form becomes an Einstein one. Next, we shall get an expression of the Riemannian curvature $R_{\omega\nu\mu\lambda}$ that does not include the tensor field $P_{\mu\lambda}$.

Let $M(H)$ be an m-dimensional l.c.K-space form with constant holomorphic sectional curvature $H$. Then we have (1.5). Transvecting (1.5) with $\gamma$ we have from the straightforward calculation

$$\Delta R_{\mu\lambda} = \{(m + 2)H + 3P\} g_{\mu\lambda} + 3(m - 4) P_{\mu\lambda},$$

where $P = \frac{1}{2} g_{\mu\lambda}$ and it can be written as

$$P = -\nabla_{\mu} \alpha_{\gamma} + \frac{1}{2} (m - 2) \|\alpha\|^2.$$

Thus we have

PROPOSITION 2.1. A 4-dimensional l.c.K-space form $M(H)$ which the tensor field $P_{\mu\lambda}$ is hybrid is an Einstein one and then the scalar field $P$ is constant.

We have from (2.2) and the Green's theorem [5].
PROPOSITION 2.2. A compact $m$-dimensional l.c.K-space form $M(H)$ which the tensor field $P_{\mu \lambda}$ is hybrid has a non-negative $P$.

Next, we shall prove the following:

THEOREM 2.3. An $m$-dimensional $(m \neq 4)$ l.c.K-space form $M(H)$ which the tensor field $P_{\mu \lambda}$ is hybrid is an Einstein one if and only if the tensor field $P_{\mu \lambda}$ is proportional to $g_{\mu \lambda}$.

PROOF. If the tensor field $P_{\mu \lambda}$ is proportional to $g_{\mu \lambda}$, then the tensor field $P_{\mu \lambda}$ can be written as

$$P_{\mu \lambda} = \frac{P}{m} g_{\mu \lambda}. \quad (2.3)$$

Thus we have from (2.1) and (2.3)

$$R_{\mu \lambda} = \{(m + 2)H + \frac{6(m - 2)P}{m}\} g_{\mu \lambda}. \quad (2.4)$$

The inverse is trivial, so we omit its proof.

COROLLARY 2.4. An $m$-dimensional $(m \neq 4)$ Einstein l.c.K-space form $M(H)$ which the tensor field $P_{\mu \lambda}$ is hybrid is a complex space form if $P = 0$.

Transvecting (2.1) with $\varepsilon_{\mu \lambda}$, we have

$$4R = m(m + 2)H + 6(m - 2)P, \quad (2.4)$$

where $R$ denotes the scalar curvature with respect to $g_{\mu \lambda}$. By virtue of (2.1) and (2.4), we can easily see that

$$R_{\mu \lambda} = R_{\mu \lambda} = \frac{4(m - 4)}{3(m - 4)} P_{\mu \lambda}. \quad (2.5)$$

Substituting (2.5) and (2.6) into (1.5), we obtain

$$R_{\mu \lambda} = \{(m - 4)H + \frac{6(m - 2)P}{m}\} g_{\mu \lambda}. \quad (2.6)$$

Thus we have

PROPOSITION 2.5. In an $m$-dimensional $(m \neq 2, 4)$ l.c.K-space form $M(H)$ which the tensor field $P_{\mu \lambda}$ is hybrid, the Riemannian curvature tensor $R_{\mu \lambda}$ can be written as

$$R_{\mu \lambda} = \frac{4}{3(m - 4)} P_{\mu \lambda}, \quad (2.7)$$

without $P_{\mu \lambda}$.

3. COMPACT L.C.K-SPACE FORMS.

In this section, we shall mainly deal with compact l.c.K-space form.

Let $M(H)$ be an $m$-dimensional l.c.K-space form with constant holomorphic sectional curvature $H$. If we assume that the scalar curvature $R$ is constant, then by virtue of (2.4) all of the scalar fields $R, H$, and $P$ are constant. Under this assumption, differentiating (2.1) covariantly, we get

$$4\nabla_{\mu} R_{\mu} = 3(m - 4)\nabla_{\mu} P. \quad (3.1)$$

Substituting (1.2) into the above equation, we have

$$4\nabla_{\mu} R_{\mu} = 3(m - 4)\nabla_{\mu} P = \{\nabla_{\mu} P - \nabla_{\mu} P - \nabla_{\mu} P - \nabla_{\mu} P - \nabla_{\mu} P\}. \quad (3.2)$$

By virtue of the Ricci identity [5] and the assumption $\nabla_{\mu} P = \nabla_{\mu} P$, the equation (3.2) implies
\[ 4\left( \nabla_{\omega} R_{\mu\nu} - \nabla_{\nu} R_{\omega\mu} \right) = 3(m - 4)(R - \frac{1}{2}\nabla \nabla g_{\mu\nu}) + \alpha_\omega (\nabla \omega) - \alpha_\nu (\nabla \omega) \]

Transvecting the above equation with \( g \) and taking account of the formula \( 2\nabla R^E = \nabla R [5] \), we obtain

\[ R_{\omega\nu}^E + (\nabla_\omega^E)_{\omega\nu} + \frac{1}{2}(m - 2)\nabla \omega = 0. \tag{3.3} \]

Substituting (2.1) into (3.3), we obtain

\[ (m + 2)H + 3\mathcal{V}^2 + \nabla \mathcal{V}^2 + \frac{m - 4}{2} \nabla \mathcal{V}^2 = 0. \tag{3.4} \]

Thus we have

**THEOREM 3.1.** In an \( m \)-dimensional (\( m \neq 2, 4 \)) l.c.K-space form \( M(H) \) which the tensor field \( P_{\mu\lambda} \) is hybrid and the scalar curvature \( R \) is constant, the length \( \mathcal{V} \) of the Lee form \( \alpha_\lambda \) is non-zero constant if and only if

\[ (m + 2)H + 3\mathcal{V}^2 + \nabla \mathcal{V}^2 = 0. \tag{3.5} \]

By virtue of (3.5) and the Green's theorem, we have

**COROLLARY 3.2.** In a compact \( m \)-dimensional (\( m \neq 2, 4 \)) l.c.K-space form \( M(H) \) which the tensor field \( P_{\mu\lambda} \) is hybrid and the scalar curvature \( R \) is constant, if the length \( \mathcal{V} \) of the Lee form \( \alpha_\lambda \) is non-zero constant, then there exists the following relation between the holomorphic sectional curvature \( H \) and the length \( \mathcal{V} \) of the Lee form \( \alpha_\lambda \):

\[ (m + 2)H + 3\mathcal{V}^2 = 0. \tag{3.6} \]

**COROLLARY 3.3.** There does not exist a compact \( m \)-dimensional (\( m \neq 2, 4 \)) l.c.K-space form \( M(H) \) which the tensor field \( P_{\mu\lambda} \) is hybrid and the holomorphic sectional curvature \( H \) is positive if the length \( \mathcal{V} \) of the Lee form \( \alpha_\lambda \) and the scalar curvature \( R \) are constant. Especially, if \( H = 0 \), then the manifold \( M \) must be locally Euclidean, that is, the Riemannian curvature tensor \( R_{\omega\nu\mu\lambda} \) is identically zero.

The following proposition was proved by T.Kashiwada [1];

**PROPOSITION 3.4.** In a compact \( m \)-dimensional (\( m \neq 2 \)) l.c.K-manifold \( M \), if

\[ R_{\mu\lambda} - R > 0 \tag{3.7} \]

holds good, then the manifold \( M \) is a Kähler manifold, where \( R_{\mu\lambda} = \frac{1}{2}\nabla_{\mu} \nabla_{\lambda} \). The inequality \( > \) in this case is naturally reduced to \( = \).

Now, let \( M(H) \) be a compact \( m \)-dimensional (\( m \neq 2, 4 \)) l.c.K-space form. Then transvecting (2.5) with \( P_{\omega\nu\mu\lambda} \), we get

\[ \frac{1}{2}R_{\omega\nu\mu\lambda} + \frac{m(m + 2)H + R}{3}. \tag{3.8} \]

By virtue of (2.4) and (3.8), we obtain

\[ H_{\mu\lambda} - R = \frac{m(m + 2)H - 4R}{3}. \tag{3.9} \]

Thus we have from PROPOSITION 3.4 and (3.9)

**THEOREM 3.5.** In a compact \( m \)-dimensional (\( m \neq 2, 4 \)) l.c.K-space form \( M(H) \) which the tensor field \( P_{\mu\lambda} \) is hybrid, if the inequality \( m(m + 2)H > 4R \) holds good, then the manifold \( M \) is a complex space form.

4. **RECURRENT L.C.K-SPACE FORMS.**

A Riemannian manifold \( M \) is said to be recurrent if the Riemannian curvature tensor
\( R_{\omega \nu \mu \lambda} \) satisfies
\[
\nabla^{\kappa} R_{\omega \nu \mu \lambda} = \theta_{\kappa}^{\rho} R_{\omega \nu \mu \lambda}
\]  
(4.1)
for a certain non-zero vector field \( \theta_{\kappa} \). For a recurrent Riemannian manifold, it is trivial that
\[
\nabla_{\lambda} R_{\omega \nu \mu \lambda} = \theta_{\lambda}^{\rho} R_{\omega \nu \mu \lambda}, \quad \nabla_{\lambda} R = \theta_{\lambda} R.
\]  
(4.2)

Now, let \( M(\mathcal{H}) \) be an \( m \)-dimensional (\( m \neq 2, 4 \)) recurrent l.c.K-space form which the tensor field \( P_{\mu \nu} \) is hybrid. Then we have (2.7) and (4.1). Differentiating (2.7) covariantly and taking account of (4.1) and (4.2), we have
\[
\frac{H}{m - 2} \left( \partial_{\kappa} \left( \eta_{\kappa} \right) \eta_{\kappa} - \eta_{\kappa} + \eta_{\kappa} \right) + \frac{(m - 4)(m - 1)H + R}{(m - 2)(m - 4)} \left( \eta_{\kappa} \right) \eta_{\kappa} - \eta_{\kappa} - \eta_{\kappa} + \eta_{\kappa} \right)
\]  
(4.3)

Transvecting (4.3) with \( P_{\mu \nu} \), we get
\[
\frac{(m + 2)H}{3} \eta_{\kappa} + \frac{(m - 4)(m - 1)H + R}{3(m - 2)} \left( \eta_{\kappa} \right) \eta_{\kappa} - \eta_{\kappa} + \eta_{\kappa} \right)
\]  
(4.4)

Thus we have

**THEOREM 4.1.** An \( m \)-dimensional (\( m \neq 2, 4 \)) recurrent l.c.K-space form \( M(\mathcal{H}) \) which the tensor field \( P_{\mu \nu} \) is hybrid is trivial, that is, the manifold is locally symmetric or of zero holomorphic sectional curvature.

Let \( M(\mathcal{H}) \) be a 4-dimensional recurrent l.c.K-space form. Then, by virtue of **PROPOSITION 2.1**, the manifold is Einstein. Thus we have from (2.1) and (4.2)
\[
(2H + P) \theta_{\kappa} = 0.
\]  
(4.5)
Thus we have

**THEOREM 4.2.** A 4-dimensional recurrent l.c.K-space form $\mathcal{M}(\mathcal{H})$ which the tensor field $P_{\mu\lambda}$ is hybrid is trivial or the manifold has a property $2\mathcal{H} + P = 0$.

**REFERENCES**


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