CONVEX CURVES OF BOUNDED TYPE

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ABSTRACT. Let C be a simple closed convex curve in the plane for which the radius of curvature ρ is a continuous function of the arc length. Such a curve is called a convex curve of bounded type, if ρ lies between two fixed positive bounds. Here we give a new and simpler proof of Blaschke's Rolling Theorem. We prove one new theorem and suggest a number of open problems.

KEY WORDS AND PHRASES. *Convex curve, bounded type, perimeter centroid, Blaschke's Rolling Theorem, parallel curves, mass distribution on a curve.*

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1. INTRODUCTION.

Let C be a simple closed convex curve in the plane. Such curves have been the subject of numerous studies[1, 2, 3, 5, 6, 7, 8, 12, 14, 15, 19, and 24] to cite only a few. Here we will refine the objects of study by looking at certain subsets. Throughout this paper C is a simple closed convex curve in the plane for which the radius of curvature ρ is a continuous function of arc length. Our refinement consists of putting upper and lower bounds on ρ .

Definition 1. We say that C is a convex curve of bounded type if there are constants R_1 and R_2 such that

$$0 < R_1 \leq \rho \leq R_2 \quad (1.1)$$

at every point of C . We let $CV(R_1, R_2)$ denote the set of all such curves that satisfy (1.1) for fixed R_1 and R_2 .

Theorems about the class $CV(R_1, R_2)$ appear in the literature (see for example Theorem 3), but as far as I am aware, this class has not been given a specific name and symbol until now. In this work we are concerned with one type of question, namely how close can C come to its "center" and how far away from its "center" can C go.

The center can be defined in various ways. For example the center of mass of the region bounded by C when the region has a uniform mass distribution. Or the center could be the center of mass of the curve C when the mass is distributed either uniformly or as some other function of s the arc length on C . In any case we can take the origin as the center of mass without loss of generality. For each fixed curve in $CV(R_1, R_2)$ set

$$D_1 = \min_{P \in C} |OP|, \quad \text{and} \quad D_2 = \max_{P \in C} |OP|. \quad (1.2)$$

Our main result is

Theorem 1. Suppose that $C \in CV(R_1, R_2)$ and the center is the center of mass of the curve C . If the mass distribution on C is uniform, then

$$R_1 \leq D_1 \leq D_2 \leq R_2. \quad (1.3)$$

The two circles of radius R_1 and R_2 show that the inequality (1.3) is sharp.

Bose and Roy [6] call this center the perimeter centroid.

In section 2 we review some facts about parallel curves and we give a new proof of Blaschke's Rolling Theorem [3, pp. 114-116]. In section 3 we prove Theorem 1. In section 4 we suggest some topics for further research on the set $CV(R_1, R_2)$.

2. PARALLEL CURVES.

Let $C \in CV(R_1, R_2)$. We select the parameter s (arc length) so that s increases as the point $P = P(s)$ traverses C in the counterclockwise direction. Let ϕ denote, as usual, the angle that the unit tangent $\underline{T}$ makes with the positive x -axis, and let $\underline{N}$ be the unit inward normal to C at the point $P = P(s)$. We recall that

$$\frac{dx}{ds} = \cos \phi, \quad \frac{dy}{ds} = \sin \phi, \quad (2.1)$$

$$\underline{T} = \frac{dx}{ds}\underline{i} + \frac{dy}{ds}\underline{j} = (\cos \phi)\underline{i} + (\sin \phi)\underline{j}, \quad (2.2)$$

and

$$\underline{N} = (-\sin \phi)\underline{i} + (\cos \phi)\underline{j}. \quad (2.3)$$

If $\underline{V} = \underline{V}(s)$ is the vector equation of C we introduce a second curve C^* defined by the vector equation $\underline{V}^* = \underline{V}(s) + A\underline{N}$, where A is a constant. The curve C^* is said to be parallel to C , see [13 pp. 80-84, 18 p. 67, and 19]. Fig. 1 shows a number of curves parallel to the ellipse $x^2/9 + y^2/4 = 1$. The curve C^* is also a Bertrand mate of C , although the term Bertrand curve usually refers to twisted curves in space [4, p. 35].

If $P(x, y)$ is a point on C and $P^*(x^*, y^*)$ is the corresponding point on the parallel curve C^* , then

$$x^* = x - A \sin \phi \quad \text{and} \quad y^* = y + A \cos \phi. \quad (2.4)$$

If $\kappa = 1/\rho$ is the curvature of C at P , then $\kappa = d\phi/ds$ and from (2.4) and (2.1)

$$\frac{dx^*}{ds} = \frac{dx}{ds} - A\kappa \cos \phi = (1-A\kappa)\cos \phi, \quad (2.5)$$

and

$$\frac{dy^*}{ds} = \frac{dy}{ds} - A\kappa \sin \phi = (1-A\kappa)\sin \phi. \quad (2.6)$$

We let s^* , κ^* , and ρ^* denote, arc length, curvature, and radius of curvature at the corresponding point on C^* . Then (2.5) and (2.6) give

$$\left(\frac{ds^*}{ds}\right)^2 = \left(\frac{dx^*}{ds}\right)^2 + \left(\frac{dy^*}{ds}\right)^2 = (1-A\kappa)^2. \quad (2.7)$$

If $R_1 < A < R_2$, then the curve C^* may have cusps as shown in Fig. 1. If $A < R_1$ we set $ds^*/ds = 1 - A\kappa > 0$. If $A > R_2$ then $1 - A\kappa < 0$ and we set $ds^*/ds = |1 - A\kappa|$.

Thus in either case s^* and s increase together. In the first case, $A < R_1$, we have

$$\begin{aligned} \underline{T}^* &\equiv \frac{d\underline{V}^*}{ds^*} = \frac{d\underline{V}}{ds} \frac{ds}{ds^*} = [(1-A\kappa)\cos \phi \underline{i} + (1-A\kappa)\sin \phi \underline{j}] \frac{1}{1-A\kappa} \\ &= (\cos \phi)\underline{i} + (\sin \phi)\underline{j} = \underline{T}. \end{aligned} \quad (2.8)$$

If $A > R_2$, then the same type computation gives $\underline{T}^* = -\underline{T}$.

Lemma 1. If $A < R_1$, then the directed tangents at corresponding points of C and C^* are parallel and point in the same direction. Further $\underline{N}^* = \underline{N}$. If $A > R_2$, then $\underline{T}^* = -\underline{T}$ and $\underline{N}^* = -\underline{N}$.

Figure 1

Lemma 2. If $C \in CV(R_1, R_2)$ and $A < R_1$, then C^* is locally convex and at corresponding points $\rho^* = \rho - A$.

By locally convex we mean $\phi^{*'}(s^*) > 0$ at each point of C^* .

Proof. From Lemma 1 we have $\phi^* = \phi$ at corresponding points. Hence, for the curvature

$$\kappa^* = \frac{d\phi^*}{ds^*} = \frac{d\phi}{ds^*} = \frac{d\phi}{ds} \frac{ds}{ds^*} = \kappa \frac{1}{1-A/\rho}. \quad (2.9)$$

Thus $\kappa^* > 0$ whenever $\kappa > 0$, and C^* is locally convex. Further

$$\rho^* = \frac{1}{\kappa^*} = \frac{1-A/\rho}{\kappa} = \rho \left(1 - \frac{A}{\rho}\right) = \rho - A. \quad (2.10)$$

Of course $\rho^* = \rho - A$ is geometrically obvious from the definition of C^* . Q.E.D.

If $A > R_2$, the factor $1/(1-A/\rho)$ in (2.9) is replaced by $\rho/(A-\rho)$. Again C^* is locally convex, but in this case $\rho^* = A - \rho$.

It is geometrically obvious that if $A < R_1$ or $A > R_2$, then C^* is a simple closed curve. It seems that a direct proof is rather elusive. The difficulty may lie in the following example. Let C^* be the image of $|z| = 1$ under the complex function $f(z) = z + z^2$. Then C^* is convex in the sense that $\kappa^* > 0$ at every point, so C^* is locally convex. But this curve fails to be simple. Nevertheless we have

Theorem 2. If $C \in CV(R_1, R_2)$ and $A < R_1$ or $A > R_2$, then C^* is a simple closed convex curve. If $A < R_1$, then $C^* \in CV(R_1^*, R_2^*)$, where

$$R_1^* = R_1 - A, \quad \text{and} \quad R_2^* = R_2 - A. \quad (2.11)$$

If $A > R_2$, then $C^* \in CV(R_1^*, R_2^*)$, where

$$R_1^* = A - R_2, \quad \text{and} \quad R_2^* = A - R_1. \quad (2.12)$$

Proof. We have already seen that C^* is locally convex, but the example shows this is not sufficient to prove that C^* is simple. On C^* let

$$\Delta\phi^* = \phi^*(L^*) - \phi^*(0) = \int_0^{L^*} \frac{d\phi^*}{ds^*} ds^*, \quad (2.13)$$

where L^* is the length of C^* . We make a change of variables from s^* to s . If $A < R_1$, then $\phi^* = \phi$ and

$$\frac{d\phi^*}{ds} = \frac{d\phi}{ds}. \quad (2.14)$$

Then (2.13) gives

$$\Delta\phi^* = \int_0^{L^*} \frac{d\phi^*}{ds} \frac{ds}{ds^*} ds^* = \int_0^L \frac{d\phi^*}{ds} ds = \int_0^L \frac{d\phi}{ds} ds = \int_0^L d\phi = 2\pi. \quad (2.15)$$

Since C^* is locally convex and $\Delta\phi^* = 2\pi$, we see that C^* is a simple curve.

If $A > R_2$, then $\phi^* = \phi + \pi$. Hence (2.14) is still true and the proof remains valid. The relations (2.11) and (2.12) follow from $\rho^* = \rho - A$ and $\rho^* = A - \rho$ respectively. Q.E.D.

Theorem 3. Let $C \in CV(R_1, R_2)$ and let K be a circle tangent internally to C at any point P_0 of C . If K has radius R_1 , then K is contained in C . If K has radius R_2 , then K contains C .

This theorem is often called Blaschke's Rolling Theorem, because it states that (a) a circle of radius R_1 can roll around the inside of C , and (b) a circle of radius R_2 can roll around the outside of C . Blaschke has extended his theorem to 3-dimensional space [3, p. 118]. For further work on this theorem, and various extensions see [11, 17, 20, and 22].

To be precise the phrase "internally tangent" means that K is tangent to C at P_0 and the center of K lies on the inward normal to C at P_0 . Thus the location of the center is given by equation set (2.4) with A replaced by R_α the radius of the tangent circle ($\alpha = 1, 2$). We say that K is contained in C if K is contained in the closure of the region bounded by C . Further K contains C , if C is in the closed disk bounded by K .

Proof of Theorem 3. We first show that the curve C cannot cross the circle K in a neighborhood of P_0 , the point of contact. Without loss of generality let P_0 be the origin and let K and C be tangent to the x -axis at the origin. Further suppose that both the circle and the curve lie above the x -axis, except at the origin. In this position the lower half of the circle will have equation

$$Y = R - \sqrt{R^2 - x^2}, \quad -R \leq x \leq R. \quad (2.16)$$

If $y = f(x)$ is the equation of C in a suitable neighborhood, $I : -\epsilon \leq x \leq \epsilon$, then we have $f'(x) \operatorname{sgn} x \geq 0$ and $f''(x) \geq 0$ in I .

Lemma 3. Suppose that $\rho \geq R$ in I , where ρ is the radius of curvature on C . Then, under the conditions described above

$$y(x) \leq Y(x) = R - \sqrt{R^2 - x^2}, \quad x \in I.$$

Thus in I , the curve C cannot cross from outside to inside K , but of course C may coincide with K . We omit the proof of Lemma 3, but it follows directly from two integrations, starting with the inequality

$$\frac{1}{\rho} = \frac{y''(x)}{[1+(y'(x))^2]^{3/2}} \leq \frac{1}{R}. \quad (2.17)$$

By reversing the inequality signs we have

Lemma 4. Suppose that $\rho \leq R$ in I . Then under the conditions on K and C described above

$$y(x) \geq Y(x) = R - \sqrt{R^2 - x^2}, \quad x \in I.$$

Thus in I , the curve C cannot cross from inside to outside K , but of course C may coincide with K .

From these two lemmas we see that if $R = R_1$ or $R = R_2$, then C cannot cross into or out of K in a neighborhood of a point of tangency. To complete the proof of Theorem 3, we must obtain this same result in the large.

First suppose that K has radius R_1 and is tangent internally to C at P_0 . If K is not contained in C , then K crosses C at a point P_2 distinct from P_0 . Then we may find a smaller circle K_0 with radius $R_0 < R_1$, and such that K_0 is tangent internally to C

at P_0 , and is tangent to C at another point P_1 , see Fig. 2.

Figure 2

Now consider the parallel curve C^* with $A = R_0 < R_1$. By Theorem 2, this curve is a simple close curve. On the other hand, the center D of the circle K_0 is at least a double point of C^* because it is the corresponding point for both P_0 and P_1 . Hence we have a contradiction.

For the second part of Theorem 3 let K be a circle with radius R_2 and tangent internally to C at P_0 . If K does not contain C , then K crosses C at a point P_2 distinct from P_0 . Then we may find a larger circle K_0 with radius $R_0 > R_2$ and such that K_0 is tangent internally to C at P_0 and is tangent to C at another point P_1 . Again consider the parallel curve C^* with $A = R_0 > R_2$. By Theorem 2 this curve C^* is a simple closed curve. Just as before we obtain a contradiction because D the center of K_0 is at least a double point on C^* . Q.E.D.

Corollary 1. Let $L(C)$ denote the length of C and let $A(C)$ denote the area of the region enclosed by C . If $C \in CV(R_1, R_2)$, then

$$2\pi R_1 \leq L(C) \leq 2\pi R_2, \quad (2.18)$$

and

$$\pi R_1^2 \leq A(C) \leq \pi R_2^2. \quad (2.19)$$

The circles of radius R_1 and R_2 show that both of these inequalities are sharp.

The inequalities (2.18) and (2.19) are well known, see for example [1, p. 352], [15], and [16, Vol. 1, pp. 529 and 548].

3. PROOF OF THEOREM 1.

Let $C \in CV(R_1, R_2)$ and let $\mu(s)$ be a mass distribution of C . We exclude the trivial case in which all of the mass is concentrated at one point. Then the center of mass will be an interior point of the region bounded by C . Without loss of generality we select the center of mass to be the origin. If L is the length of C , then

$$\int_0^L x(s)\mu(s)ds = 0, \quad \text{and} \quad \int_0^L y(s)\mu(s)ds = 0 \quad (3.1)$$

Now consider the parallel curve C^* where $A < R_1$, and let $\mu^* = \mu^*(s^*)$ be a mass distribution on C^* . Then the moments M_{x^*} and M_{y^*} are given by

$$M_{y^*} = \int_0^L x^*(s^*) \mu^*(s^*) ds^*, \quad (3.2)$$

and

$$M_{x^*} = \int_0^L y^*(s^*) \mu^*(s^*) ds^*. \quad (3.3)$$

The change of variable from s^* to s yields

$$M_{y^*} = \int_0^L (x - A \frac{dy}{ds}) \mu^*(s^*(s)) (1 - \frac{A}{\rho(s)}) ds, \quad (3.4)$$

and

$$M_{x^*} = \int_0^L (y + A \frac{dx}{ds}) \mu^*(s^*(s)) (1 - \frac{A}{\rho(s)}) ds. \quad (3.5)$$

We now specialize, by setting $\mu(s) = 1$ on C and selecting $\mu^*(s^*)$ so that

$$\mu^*(s^*(s)) = \frac{1}{1 - A/\rho(s)} > 0. \quad (3.6)$$

Then (3.4) and (3.5) give

$$M_{y^*} = \int_0^L (x - A \frac{dy}{ds}) ds = \int_0^L x ds - A \int_C dy = 0,$$

and

$$M_{x^*} = \int_0^L (y + A \frac{dx}{ds}) ds = \int_0^L y ds + A \int_C dx = 0,$$

from (3.1). Thus with the mass distribution (3.6), the center of mass of C^* is also at the origin. Since C^* is a simple closed convex curve, the origin lies inside C^* and hence $D_1 \geq A$. Finally we note that A may be taken arbitrarily close to $R_1 \equiv \min |\rho|$ for points on C . Therefore

$$D_1 \geq R_1. \quad (3.7)$$

To prove that $D_2 \leq R_2$, we consider the parallel curve C^* where now $A > R_2$. For this curve equations (3.2) and (3.3) still hold. However, in this case we have

$$\frac{ds^*}{ds} = \frac{A}{\rho} - 1 > 0. \quad (3.8)$$

Thus in equations (3.4) and (3.5) we must replace the factor $1 - A/\rho$ by $A/\rho - 1$. If we select $\mu(s) = 1$ on C and μ^* on C^* so that

$$\mu^*(s^*(s)) = \frac{\rho(s)}{A - \rho(s)} > 0, \quad (3.9)$$

then this mass distribution will give $M_{x^*} = M_{y^*} = 0$. By Theorem 2, the curve C^* is a simple closed convex curve and the origin which is also the center of mass lies inside the region bounded by C^* .

Now let P be a point on C furthest from the origin. Then OP is normal to C at P . If P^* is the point on C^* corresponding to P , then PP^* is also normal to C at P . Hence the points P , O , and P^* are collinear.

Finally we observe that by Lemma 1, the directed tangents to C and C^* at the points P and P^* have opposite directions. Hence the origin is an interior point of the line segment PP^* . Therefore, $|OP| < |PP^*| = A$. Since A may be taken arbitrarily close to R_2 , we have $D_2 \leq R_2$. Q.E.D.

4. FURTHER QUESTION FOR STUDY.

We observe that the inequality $R_1 \leq D_1 \leq D_2 \leq R_2$ is sharp for the circles of radius R_1 and R_2 . But in each extreme case ρ does not vary throughout the interval $[R_1, R_2]$ but instead is a constant at one end of the interval. The question naturally

arises, can we find better bounds for D_1 and D_2 . ρ is a continuous function of s whose values fill out the interval $[R_1, R_2]$. A first candidate for consideration is the ellipse $x = a \cos t$, $y = b \sin t$, $0 \leq t \leq 2\pi$, with $0 < b < a$. If we set

$$b = (R_1^2 R_2^2)^{\frac{1}{3}}, \quad \text{and} \quad a = (R_1 R_2^2)^{\frac{1}{3}}, \quad (4.1)$$

then ρ fills out the interval $[R_1, R_2]$. Further $D_1 = b$ and $D_2 = a$, so the expressions in (4.1) may appear as the proper lower and upper bounds for D . If true, this would improve the bounds R_1 and R_2 given in Theorem 1. However, by piecing together arcs of circles, we can show that no better bounds than $R_1 \leq D_1 \leq D_2 \leq R_2$ can be obtained. To see this, we give C only in the first quadrant and complete the curve by reflecting C in the x - and y -axes.

Let C_1 and C_2 be the two arcs defined by

$$\begin{aligned} x &= a + R_1 \cos t, & x &= R_2 \cos t, \\ y &= R_1 \sin t, & y &= -b + R_2 \sin t, \\ 0 &\leq t \leq T, & T &\leq t \leq \pi/2, \end{aligned} \quad (4.2)$$

respectively. The endpoints of the two arcs will meet when $t = T$ if we select $a = (R_2 - R_1) \cos T$ and $b = (R_2 - R_1) \sin T$ where $0 < R_1 < R_2$. If we compute the first derivative for the two arcs at $t = T$, they will not be equal, but the tangent vectors will be parallel, so that for this choice of a and b , the curve $C = C_1 \cup C_2$ is a smooth curve. Further $\rho = R_1$ on C_1 and $\rho = R_2$ on C_2 . Finally $D_1 = R_1 \sin T + R_2(1 - \sin T)$ and $D_1 \rightarrow R_1$ as $T \rightarrow \pi/2$. Similarly $D_2 = R_2 \cos T + R_1(1 - \cos T)$ and $D_2 \rightarrow R_2$ as $T \rightarrow 0$. Thus no better bounds than $D_2 \leq R_2$ and $D_1 \geq R_1$ can be proved under the hypotheses stated. Of course ρ is not continuous in a neighborhood of $P(T)$, where C_1 and C_2 meet, but it is merely a matter of labor to alter the curve slightly at $P(T)$ to make ρ continuous.

Perhaps some better bounds for D_1 and D_2 can be obtained if we impose a further restriction that the average values of ρ over the curve be a fixed number such as $(R_1 + R_2)/2$.

One can also examine the problem of finding sharp bounds for D_1 and D_2 if the mass distribution has some fixed pattern, other than uniform. For example, Steiner [23], and [24, pp. 99-159] has considered curves in which the mass distribution on C is proportional to the curvature at each point of C . More generally one can select the mass distribution to be some other function of $\kappa = 1/\rho$.

One can also consider Theorem 1, when the center of mass of C is replaced by the center of mass of the region enclosed by C . With this replacement, Theorem 1 was proved earlier by Nikliborc [21] and Blaschke [2]. It is reasonably clear that the center of mass of a curve C is in general different from the center of mass of the region enclosed by C , but it may be of interest to examine a particular example.

Let C be $f(|z|=1)$ under $f(z) = z + az^2$, where $0 < a < 1/4$. Then C is symmetric with respect to the x -axis and if the mass distribution is uniform on C then the center of mass will be on the x -axis. Hence it suffices to compute the x -coordinate. Let $\tilde{x}_d$ and $\tilde{x}_c$ denote this coordinate for the domain center of mass and the curve center of mass respectively. As easy computation gives

$$\tilde{x}_d = \frac{a}{1+2a^2}. \quad (4.3)$$

A somewhat longer computation gives

$$\tilde{x}_C = \frac{M_Y}{L}, \quad (4.4)$$

where

$$L = \int_0^{2\pi} \sqrt{1+4a \cos \theta + 4a^2} \, d\theta, \quad (4.5)$$

and

$$M_Y = \int_0^{2\pi} (\cos \theta + a \cos 2\theta) \sqrt{1+4a \cos \theta + 4a^2} \, d\theta. \quad (4.6)$$

Hence

$$\tilde{x}_C = a(1 - \frac{5}{2}a^2 + \dots).$$

It is clear that in general $\tilde{x}_d \neq \tilde{x}_C$.

We may distinguish a third center of mass $\tilde{x}_s$, which we will call the conformal strip center. Suppose that $f(z)$ maps E conformally onto D , with $f(0) = 0$. Set $\tilde{x}_s(r, 1)$ the x -coordinate of the center of mass of the strip bounded by the curves $f(|z|=1)$ and $f(|z|=r)$, where $r < 1$. Then by definition

$$\tilde{x}_s = \lim_{r \rightarrow 1^-} \tilde{x}_s(r, 1). \quad (4.7)$$

An easy computation shows that if $f(z) = z + az^2$, $0 < a < 1/4$, and the mass distribution is uniform, then

$$\tilde{x}_s = \frac{2a}{1+4a^2}. \quad (4.8)$$

In this case $\tilde{x}_s \neq \tilde{x}_d$ unless $a = 0$. Further it is clear that in general $\tilde{x}_s \neq \tilde{x}_C$. This example suggests the problem of finding

$$\max |\tilde{x}_j - \tilde{x}_k|, \quad (4.9)$$

when C varies over the set $CV(R_1, R_2)$ and $j, k \in \{d, C, s\}$.

For other relations among various centers of mass, see Guggenheimer [10], and Kubota [19].

A computation, using $|z| = 1$ and

$$\rho = \frac{|zf'(z)|}{\operatorname{Re}(1 + zf''(z)/f'(z))}, \quad (4.10)$$

shows that for $0 < a < 1/4$, the function $f(z) = z + az^2$ gives a convex curve $f(|z|=1)$ for which the radius of curvature is

$$\rho = \frac{(1+4a \cos \theta + 4a^2)^{3/2}}{1+6a \cos \theta + 8a^2}. \quad (4.11)$$

Extreme values of ρ occur when $\theta = 0$, $\theta = \pi$, and $\cos \theta = 2a$. Thus $z + az^2$ is in $CV(R_1, R_2)$ for

$$R_1 = \sqrt{1-4a^2}, \quad \text{and} \quad R_2 = (1-2a)^2/(1-4a).$$

One can also investigate the properties of normalized univalent functions that map the unit disk conformally onto a region bounded by a curve in $CV(R_1, R_2)$. Some elementary results in this direction have been obtained by the author [9].

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