A SINGULAR BOUNDARY VALUE PROBLEM FOR NEUTRAL EQUATIONS

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ABSTRACT. Using some previous results of the author and Kirk-Schöneberg, theorems for the existence and uniqueness of an absolutely continuous solution of a singular boundary value problem for neutral equations have been proved.

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1. INTRODUCTION.

The aim of the present paper is to obtain conditions for the existence and uniqueness of an absolutely continuous solution of a singular boundary value problem for neutral equations, using the results due to Angelov [1] and Kirk-Schöneberg [2].

Let us state the singular boundary value problem: we look for an absolutely continuous solution $y(t)$ of the initial value problem

$$\begin{aligned} y'(t) &= X(\mu, t, [y(\Delta_i(t))]_{i=1}^m, [y'(\tau_l(t))]_{l=1}^n), \quad t > 0 \\ y(t) &= \psi(t, \mu), \quad y'(t) = \frac{\partial \psi(t, \mu)}{\partial t}, \quad t \leq 0 \end{aligned} \quad (1.1)$$

which satisfies the condition $\lim_{t \rightarrow \infty} y(t) = y_\infty$, where y_∞ is given. Here the unknown function $y(t)$ and the parameter μ take values in some Banach space B with a norm $\|\cdot\|$ and

$$[y(\Delta_i(t))]_{i=1}^m = (y(\Delta_1(t)), \dots, y(\Delta_m(t))),$$

$$[y'(\tau_l(t))]_{l=1}^n = (y'(\tau_1(t)), \dots, y'(\tau_n(t))).$$

We shall use the standard denotations:

$$R^1 = (-\infty, \infty); \quad R_+^1 = [0, \infty); \quad R_-^1 = (-\infty, 0].$$

Let us set $x(t) = y'(t)$ for $t > 0$ and $\varphi(t, \mu) = \frac{\partial \psi(t, \mu)}{\partial t}$ for $t \leq 0$. Then we have

$$x(t) = X(\mu, t, [\psi(0, \mu) + \int_0^{\Delta_i(t)} x(s) ds]_{i=1}^m, [x(\tau_1(t))]_{l=1}^n), \quad t > 0 \quad (1.2)$$

$$x(t) = \varphi(t, \mu), \quad t \leq 0.$$

The integrals are in a Bochner's sense. (Hille-Phillips [3]).

The method used by Angelov [1] is not immediately applicable to the singular boundary value problem for an equation (1.1). In order to illustrate this fact we shall consider the following particular case:

$$y'(t) = A(t)\mu + X(\mu, t, [y(\Delta_i(t))]_{i=1}^m, [y'(\tau_1(t))]_{l=1}^n) \quad (1.3)$$

which occurs in the applications. Here $A(t)$ is a family of linear continuous operators. We must seek a solution of (1.3) in the space $L^1(R^1, B)$ which implies $\|A(t)\mu\| \in L^1(R^1_+)$. But then the supposition for the existence of $A^{-1}(t)$, (Seidov [4]), is not natural, because one of the conditions for the existence of an inverse operator is $\|A(t)\mu\| \geq m \|\mu\|$ (Kantorovich-Akilov [5], p. 209).

That is why we shall apply some recent results of Kirk-Schöneberg [2] for the operator equation in μ .

Let us formulate some auxiliary propositions due to Angelov [1].

If B_i ($i = 1, 2$) are Banach spaces with norms $\|\cdot\|_i$, then:

PROPOSITION 1 [1]. Let the following conditions hold:

1. The nonlinear continuous operators $N_i: B_1 \times B \rightarrow B_i$ satisfy the inequalities

$$\|((1+\lambda\gamma-\lambda\alpha)x + \lambda N_i(x, \mu)) - ((1+\lambda\gamma-\lambda\alpha)y + \lambda N_i(y, \mu))\|_i \geq \|x-y\|_i \quad (i=1, 2) \text{ for every } x, y \in B_i, \lambda > 0, \mu \in B, \text{ for some } \gamma > 0, \alpha > 0.$$

2. RESULTS.

The linear map $j: B_1 \rightarrow B_2$ satisfies the condition $j(N_1(x, \mu)) = N_2(jx, \mu)$ for every $x \in B_1$ and $\mu \in B$.

Then for maps $x(\mu): B \rightarrow B_1$ and $y(\mu): B \rightarrow B_2$ connected by the condition $jx(\mu) = N_2(y(\mu), \mu) + \gamma y(\mu)$ there exists a unique map $z(\mu): B \rightarrow B_1$ for which $N_1(z(\mu), \mu) + \gamma z(\mu) = x(\mu)$ and $jz(\mu) = y(\mu)$.

PROPOSITION 2 [1]. Let the conditions of Proposition 1 hold, let the $x(\mu): B \rightarrow B_1$, $y(\mu): B \rightarrow B_2$ be continuous and let $\|N_i(x, \mu) - N_i(y, \mu)\|_i \leq M\|x-y\|_i$ ($i = 1, 2$), where $\gamma > M > 0$.

Then the map $z(\mu): B \rightarrow B_1$, satisfying the equation $N_1(z(\mu), \mu) + \gamma z(\mu) = x(\mu)$ is continuous.

PROPOSITION 3 [1]. Let the conditions of Proposition 1 hold and let $z_i(\mu)$, ($i = 1, 2$) satisfy the equalities $N_1(z_i(\mu), \mu) + \gamma z_i(\mu) = x_i(\mu)$, $jz_i(\mu) = y_i(\mu)$, $N_2(y_i(\mu), \mu) + \gamma y_i(\mu) = jx_i(\mu)$. Then $\|z_1(\mu) - z_2(\mu)\|_1 \leq \frac{1}{\alpha} \|x_1(\mu) - x_2(\mu)\|_1$.

We shall formulate the results of Kirk-Schöneberg [2] in order to apply them to our problem.

Let X and Y be complete metric spaces. Recall that: 1) Y is said to be metrically convex (Menger [6]) if for all, $u, v \in Y$ with $u \neq v$ there exists $w \in Y$, $w \neq u$, $w \neq v$, such that $\rho(u, v) = \rho(u, w) + \rho(w, v)$; 2) a mapping $T: X \rightarrow Y$ is said to be closed if $u_n \rightarrow u \in X$ and $T(u_n) \rightarrow z \in Y$ imply $T(u) = z$.

LEMMA [2]. Let $T: X \rightarrow Y$ be a closed mapping. Fix $y \in Y$. Let U be an open subset of X for which $T(U)$ is open in Y , and suppose

$$\rho(Tu, Tv) > \rho(u, v)$$

for all $u, v \in U$. Suppose also that there exists $u_0 \in U$ such that

$$\rho(Tu_0, y) < \rho(Tu, y) + \rho(Tu_0, Tu)$$

for all $u \in U$.

Then there exists $u \in U$ such that $Tu = y$.

If T is a mapping of X into Y and D is an open subset of X , then T is said to be locally expansive on D if for all $u_0 \in D$ there is a neighbourhood N of u_0 contained in D such that $\rho(Tu, Tv) > \rho(u, v)$ for all $u, v \in N$.

PROPOSITION 4 [2]. Let $D \subset X$ be open, and suppose $T: X \rightarrow Y$ is a closed mapping which is locally expansive on D . Suppose also that T maps open subsets of D on to open subsets of Y , and there exists $x_0 \in D$ such that $\rho(Tx_0, y) < \rho(Tx, y)$ for all $x \in X \setminus D$. Then $y \in T(D)$.

COROLLARY [2]. Let $T: X \rightarrow Y$ be a closed mapping, which maps open subsets of X onto open subsets of Y . Suppose also that T is locally expansive on all of X . Then $T(X) = Y$.

THEOREM 1. Let the following conditions hold:

1. the functions $\Delta_1(t)$, $\tau_1(t): \mathbb{R}_+^1 \rightarrow \mathbb{R}^1$ are measurable, $\tau_1(t)$ have the property (S) (Angelov, [1]) and $\int_0^\infty \|f(\tau_1(t))\| dt < k \int_{-\infty}^\infty \|f(t)\| dt$ for some

$k = \text{const} > 0$ and every $f \in L^1(\mathbb{R}^1; B)$.

2. the function $X(u, t, u_1, \dots, u_m, v_1, \dots, v_n): B \times \mathbb{R}_+^1 \times B^{m+n} \rightarrow B$ satisfies the Caratheodory condition and

$$\|X(u, t, u_1, \dots, u_m, v_1, \dots, v_n)\| \leq \frac{1}{Y} [\alpha(u, t, \|u_1\|, \dots, \|u_m\|) + \alpha_0 \sum_{l=1}^n \|v_l\|]$$

$$\|X(u, t, u_1, \dots, u_m, v_1, \dots, v_n) - X(u, t, \bar{u}_1, \dots, \bar{u}_m, \bar{v}_1, \dots, \bar{v}_n)\| \leq$$

$$\frac{m}{Y} [\beta(u, t, \|u_1 - \bar{u}_1\|, \dots, \|u_m - \bar{u}_m\|) + \sum_{l=1}^n \beta_l \|v_l - \bar{v}_l\|]$$

for some constants $\gamma > m > 0$, $\alpha_0, \beta_1 > 0$; $\alpha, \beta: B \times R_+^{m+1} \rightarrow R_+^1$ are comparison functions (Angelov [1]); $\alpha(\mu, \dots, u_1, \dots, u_m), \beta(\mu, \dots, u_1, \dots, u_m) \in L^1(R_+^1)$ and

$$\int_0^\infty \beta(\mu, t, v, \dots, v) dt + k v \sum_{l=1}^n \beta_l \leq v, \quad v \in R_+^1.$$

Besides, $X(\mu, t, \dots)$ is uniformly continuous in μ with respect to the other variables (Angelov [1]).

3. the initial function $\varphi(\cdot, \mu) \in L^1(R_-^1; B)$ for every $\mu \in B$ and

$$\lim_{\mu \rightarrow \mu_0} \int_{-\infty}^0 \|\varphi(t, \mu) - \varphi(t, \mu_0)\| dt = 0.$$

Then there exists a unique solution $x(\cdot, \mu) \in L^1(R^1; B)$ of the initial value problem (2), which depends continuously on μ .

Proof: Let B_1 be the Banach space $L^1(R^1; B)$ with norm $\|f\| = \int_{-\infty}^{+\infty} \|f(t)\| dt$ and B_2 be the Banach space $L^1(R_-^1; B)$ with norm $\|g\|_2 = \int_{-\infty}^0 \|g(t)\| dt$.

Define the operators $N_i: B_1 \times B \rightarrow B_i$ ($i = 1, 2$)

$$N_1(f, \mu)(t) = \begin{cases} -\gamma X(\mu, t, [\psi(0, \mu) + \int_0^{\Delta_1(t)} f(s) ds]_{i=1}^m), \\ [f(\tau_1(t))]_{l=1}^n, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$f \in B_1$, $\mu \in B$ and $N_2(g, \mu)(t) = 0$, $t \leq 0$, $g \in B_2$, $\mu \in B$.

The map $j: B_1 \rightarrow B_2$ is defined as in Theorem 1 (Angelov [1]).

Since $N_1(f, \mu)(t)$ is strongly measurable, then the inequalities

$$\begin{aligned} \|N_1(f, \mu)(t)\| &\leq \alpha(\mu, t, \|\psi(0, \mu)\| + \|f\|_1, \dots, \|\psi(0, \mu)\| + \|f\|_1) \\ &\quad + \alpha_0 \sum_{l=1}^n \|f(\tau_1(t))\| \end{aligned}$$

show that $f \in B_1$ implies $N_1(f, \mu) \in B_1$ for each $\mu \in B$.

We are going to show the Lipschitz continuity of the operator N_1 :

$$\begin{aligned} \int_0^\infty \|N_1(f, \mu)(t) - N_1(g, \mu)(t)\| dt &\leq m \int_0^\infty \beta(\mu, t, \|f-g\|_1, \dots, \|f-g\|_1) dt \\ &\quad + \sum_{l=1}^n \beta_l \int_0^\infty \|f(\tau_1(t)) - g(\tau_1(t))\| dt \\ &\leq m \int_0^\infty \beta(\mu, t, \|f-g\|_1, \dots, \|f-g\|_1) dt + k \|f-g\|_1 \sum_{l=1}^n \beta_l \leq m \|f-g\|_1. \end{aligned}$$

Further on the proof is analogous to the one of Theorem 1 from Angelov[1].

THEOREM 2. Let the conditions of Theorem 1 hold and let $x(\varphi_i, \mu)(t)$, ($i = 1, 2$) be solutions of the problem (2) with initial functions $\varphi_i(t, \mu)$. Then

$$\int_{-\infty}^{\infty} \|x(\varphi_1, \mu)(t) - x(\varphi_2, \mu)(t)\| dt < \frac{\gamma}{\gamma - m} \int_{-\infty}^0 \|\varphi_1(t, \mu) - \varphi_2(t, \mu)\| dt.$$

THEOREM 3. Let the conditions of Theorem 1 hold true. If, in addition, we suppose:

1. $\psi(0, \cdot)$ is locally expansive, closed and maps open subsets of B onto open subsets of B .
2. for every $\mu \in B$ there is a neighbourhood U of μ such that

$$\|X(\mu_1, t, \dots) - X(\mu_2, t, \dots)\| > \xi(t) \|\mu_1 - \mu_2\|$$

for $\mu_1, \mu_2 \in U$, where $\xi(t) \in L^1(\mathbb{R}_+^1)$, $\xi(t) > 0$ and the mapping

$Q(\mu) = \int_0^\infty X(\mu, t, u_1, \dots, u_n) dt$ is closed and maps open subsets of B onto open subsets of B , provided the last integral exists.

Then there exists a value of the parameter μ_0 such that $x(t, \mu_0)$ is a solution of the problem (2) and the solution $y(t, \mu_0)$ of (1) satisfies the condition $\lim_{t \rightarrow \infty} y(t, \mu_0) = y_\infty$.

Proof: Let B_1 be the Banach space $L^1(\mathbb{R}_+^1; B)$ and B_2 be the Banach space $L^1(\mathbb{R}_-^1; B)$.

Define the operators $N_i: B_i \times B \rightarrow B_i$ ($i = 1, 2$), $T: B \times B_1 \rightarrow B$:

$$N_1(f, \mu)(t) = \begin{cases} -\gamma X(\mu, t, [\psi(0, \mu) + \int_0^{\Delta_1(t)} f(s) ds]_{i=1}^m), \\ [f(\tau_1(t))]_{i=1}^n, & t > 0 \\ 0, & t < 0 \end{cases}$$

where $f \in B_1$, $\mu \in B$; $N_2(g, \mu)(t) = 0$, $t < 0$, $g \in B_2$, $\mu \in B$.

$$T(\mu, f) = y_\infty - \psi(0, \mu) - \int_0^\infty X(\mu, t, [\psi(0, \mu) + \int_0^{\Delta_1(t)} f(s) ds]_{i=1}^m, [f(\tau_1(t))]_{i=1}^n) dt, \quad f \in B_1.$$

Proposition 1 implies an existence of a unique function $x(t, \mu)$ which satisfies (2) and depends continuously on μ .

On the other hand, it is easy to verify that the operator $T(u, f)$ satisfies all conditions of Corollary. Indeed, $T(u, f)$ is closed and maps open subsets of B onto open subsets of B . Besides, for every $u \in B$ there is a neighbourhood U of u such that

$$\|T(u_1, f) - T(u_2, f)\| \geq c_\psi \|u_1 - u_2\| + \int_0^\infty \xi(t) dt \|u_1 - u_2\|$$

for all $u_1, u_2 \in U$. Consequently, there is $u_0 \in B$ such that $T(u_0, f) = 0$.

So we obtain

$$\lim_{t \rightarrow \infty} y(t, u_0) = \psi(0, u_0) + \int_0^\infty x(s, u_0) ds =$$

$$\psi(0, u_0) + \int_0^\infty X(u_0, s, [\psi(0, u_0) + \int_0^{\Delta_1(s)} x(\theta, u_0) d\theta]_{i=1}^m,$$

$$[x(\tau_1(s))]_{i=1}^I ds = \psi(0, u_0) + y_\infty - \psi(0, u_0) = y_\infty.$$

Theorem 3 is thus proved.

Finally we shall note that an analogous result can be formulated for systems of the type:

$$y'_i(t) = X_i(u_i, t, [y_1(\Delta_{1k}(t))]_{k=1}^{m_1}, \dots, [y_s(\Delta_{sk}(t))]_{k=1}^{m_s},$$

$$[y'_1(\tau_{1k}(t))]_{k=1}^{n_1}, \dots, [y'_s(\tau_{sk}(t))]_{k=1}^{n_s}), \quad t > 0$$

$$y_i(t) = \psi_i(t, u_i), \quad y'_i(t) = \frac{\partial \psi_i(t, u_i)}{\partial t}, \quad t < 0$$

$$(i = 1, 2, \dots, s).$$

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