A REPRESENTATION OF JACOBI FUNCTIONS

E. Y. DEEBA
Department of Applied Mathematical Sciences
University of Houston-Downtown
Houston, Texas 77002

and

E. L. KOH
Department of Mathematics & Statistics
University of Regina
Regina, Canada S4S 0A2

(Received August 1, 1985)

Abstract: Recently, the continuous Jacobi transform and its inverse are defined and studied in [1] and [2]. In the present work, the transform is used to derive a series representation for the Jacobi functions $P_{\lambda}^{(\alpha,\beta)}(x)$, $-\frac{1}{2} \leq \alpha, \beta \leq \frac{1}{2}$, $\alpha + \beta = 0$, and $\lambda \geq -\frac{1}{2}$. The case $\alpha = \beta = 0$ yields a representation for the Legendre functions and has been dealt with in [3]. When $\lambda$ is a positive integer $n$, the representation reduces to a single term, viz., the Jacobi polynomial of degree $n$.

KEY WORDS AND PHRASES: Jacobi functions, Jacobi transform, representation, special functions.


1. Introduction. The continuous Jacobi transform and its inverse were introduced and studied in [1] and [2]. These transforms generalize the work of Butzer, Stens and Wehrens [3] on the continuous Legendre transform and the work of Debnath [4] on the discrete Jacobi transform. In [2] an application to sampling technique was given. In the present work, the continuous Jacobi transform is used to derive a series representation of Jacobi functions $P_{\lambda}^{(\alpha,\beta)}(x)$. The representation includes that for the Legendre function given in [3]. When $\lambda$ is a positive integer, the representation reduces to the Jacobi polynomial (see e.g. [5]).
2. Preliminaries. In this section we review material needed in the development of the paper.

For \( \alpha, \beta > -1, \lambda \in \mathbb{R}, \lambda + \alpha + \beta \neq 0, -1, -2, \ldots \) and \( x \in (-1, 1] \), the Jacobi function of the first kind, \( P_{\lambda}^{(\alpha, \beta)}(x) \), is given by

\[
P_{\lambda}^{(\alpha, \beta)}(x) = \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1) \Gamma(\alpha + 1)} F(-\lambda, \lambda + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2})
\]  

(2.1)

(see [6]) where

\[
F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \quad |z| < 1,
\]

\( a, b, c \) real numbers with \( c \neq 0, -1, -2, \ldots \).

Since \( P_{\lambda}^{(\alpha, \beta)}(x) = \frac{\Gamma(\lambda - \alpha + 1) \Gamma(\lambda - \alpha - \beta)}{\Gamma(1 - \lambda) \Gamma(1 - \lambda + \beta)} P_{\lambda - \alpha - \beta - 1}(x) \), we may restrict ourselves to \( \lambda \geq \frac{\alpha + \beta + 1}{2} \). The function \( P_{\lambda}^{(\alpha, \beta)}(x) \) satisfies the following relations:

\[
d \frac{d}{dx} w(x)(1-x^2) \frac{d}{dx} P_{\lambda}^{(\alpha, \beta)}(x) = -\lambda(\lambda + \alpha + \beta + 1) w(x) P_{\lambda}^{(\alpha, \beta)}(x)
\]  

(2.2)

\[
P_{\lambda}^{(\alpha, \beta)}(1) = \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1) \Gamma(\alpha + 1)},
\]

(2.3)

and

\[
(1-x^2) \frac{d}{dx} P_{\lambda}^{(\alpha, \beta)}(x) = \left( \frac{\lambda(\alpha - \beta)}{2\lambda + \alpha + \beta} - \lambda x \right) P_{\lambda}^{(\alpha, \beta)}(x)
\]  

\[+ \frac{2(\lambda + \alpha)(\lambda + \beta)}{2\lambda + \alpha + \beta} P_{\lambda - 1}^{(\alpha, \beta)}(x).
\]

(2.4)

For a proof of (2.2), (2.3) and (2.4) see [1]. The term \( w(x) \) in (2.2) is the weight function \( w(x) = (1-x)^\alpha(1+x)^\beta \) and will be used throughout the paper. Furthermore, it was shown in [1] that for \( \lambda \geq -\frac{\alpha + \beta + 1}{2} \) and for any \( x \in (-1, 1] \),

\[
|P_{\lambda}^{(\alpha, \beta)}(x)| \leq \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1) \Gamma(\alpha + 1)} + M(\lambda, \alpha, \beta) \log \frac{2}{1+x}
\]

(2.5)

where \( M(\lambda, \alpha, \beta) \) is some constant depending upon \( \lambda, \alpha \) and \( \beta \); and for any \( \lambda, \nu \geq -\frac{\alpha + \beta + 1}{2}, \lambda \neq \nu, \lambda \neq -\nu + \alpha + \beta + 1, \alpha > -\frac{1}{2}, -\frac{1}{2} < \beta < \frac{1}{2} \) we have the relation

\[
\frac{1}{2\alpha + \beta + 1} \int_{-1}^{1} w(x) P_{\lambda}^{(\alpha, \beta)}(x) P_{\nu}^{(\beta, \alpha)}(-x) dx
\]

\[= \frac{\Gamma(\lambda + \alpha + 1) \Gamma(\nu + \beta + 1)}{\Gamma(\lambda + \nu) \Gamma(\lambda + \alpha + \beta + 1)} \left\{ \frac{\sin \pi \lambda}{\Gamma(\nu + 1) \Gamma(\lambda + \alpha + \beta + 1)} - \frac{\sin \pi \nu}{\Gamma(\lambda + 1) \Gamma(\nu + \alpha + \beta + 1)} \right\}.
\]
We shall denote, throughout, the weighted square integrable functions on \((-1,1)\) by \(L_w^2(-1,1)\). For \(f \in L_w^2(-1,1)\), \(\lambda > -\frac{1}{2}, -\frac{1}{2} < \beta < \frac{1}{2}\), the continuous Jacobi transform (see [1]) is defined by

\[
\hat{f}(\alpha, \beta)(\lambda) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) P_{\lambda}^{(\alpha, \beta)}(x) f(x) \, dx
\]  

(2.7)

When \(\alpha = \beta = 0\), \(\hat{f}(\alpha, \beta)\) reduces to the continuous Legendre transform studied in [3] and when \(\lambda = n \in \mathbb{P}\) (\(\mathbb{P}\), the set of non-negative integers), \(\hat{f}(\alpha, \beta)\) reduces to the discrete Jacobi transform of Debnath [4].

It was shown in [1] that if \(1/2 f (\epsilon, \theta) (\lambda - 1/2) \in L^1(\mathbb{R}^+)\) and if \(\alpha = \beta = 0\) then for almost every \(x \in (-1,1)\), we obtain the inversion formula

\[
f(x) = 4 \int_0^{\infty} \hat{f}(\alpha, \beta) (\lambda - 1/2) P_{\lambda - 1/2}^{(\alpha, \beta)} (-x) H_0(\lambda) \lambda \sin \pi \lambda d\lambda
\]  

(2.8)

where

\[
H_0(\lambda) = \frac{\Gamma(\lambda + 1/2) }{\Gamma(\lambda + \alpha + 1/2) \Gamma(\lambda + \beta + 1/2)}.
\]

Since we needed the condition \(\alpha + \beta = 0\) to derive (2.8), we shall, from now on, assume this condition on \(\alpha\) and \(\beta\).

In [2] the second continuous Jacobi transform was studied. For \(\lambda - \beta + \epsilon f \in L^1(\mathbb{R}^+)\), it is given by

\[
\hat{f}(\alpha, \beta) (x) = 4 \int_0^{\infty} f(\lambda) P_{\lambda - 1/2}^{(\beta, \alpha)} (-x) \frac{\Gamma(\lambda + 1/2) }{\Gamma(\lambda + \beta + 1/2)} \lambda \sin \pi \lambda d\lambda
\]  

(2.9)

and the associated inversion formula is

\[
f(\lambda) = \frac{\Gamma(\lambda + 1/2) }{\Gamma(\lambda + \alpha + 1/2) \Gamma(\lambda + \beta + 1/2)} \int_{-1}^{1} w(x) P_{\lambda - 1/2}^{(\alpha, \beta)} (x) \hat{f}(\alpha, \beta)(x) \, dx
\]  

(2.10)

The relation between the different transforms (see [2]) is

\[
(\hat{f}(\alpha, \beta)(\lambda)) \hat{f}(\alpha, \beta)(\lambda) = \frac{2 \Gamma(\lambda + 1/2) }{\Gamma(\lambda + 1/2)} f(\lambda)
\]

and

\[
(\frac{\Gamma(\lambda + 1/2) }{\Gamma(\lambda + \alpha + 1/2) \Gamma(\lambda + \beta + 1/2)}) \hat{f}(\alpha, \beta)(\lambda) \hat{f}(\alpha, \beta)(\lambda) = f(\lambda)
\]

As an application of (2.9) and (2.10), it was shown in [2] that if \(f \in C(\mathbb{R}^+)\) is given by

\[
F(\lambda) = \frac{1}{2} \int_{-1}^{1} w(x) f(x) P_{\lambda - 1/2}^{(\alpha, \beta)} (x) \, dx
\]
for some $\nu > 0$, $f \in L^2_w(-1,1)$, then for all $\lambda \in \mathbb{R}^+$, we have

$$F(\lambda) = \Gamma(n+1) \prod_{n=0}^{\infty} \frac{(2n+1)\Gamma(n+1)\Gamma(\mu+\alpha+n)\sin(\mu-(n+\frac{1}{2}))}{\pi(\lambda^2 + (n+\frac{1}{2})^2)\Gamma(n+\alpha+1)\Gamma(\mu+\frac{1}{2})} \mathcal{F}(\lambda)^{n+\frac{1}{2}}. \tag{2.11}$$

We will employ (2.7), (2.8), (2.9) and (2.10) to derive the representation formula of the Jacobi functions. Since $\alpha + \beta = 0$, we shall write $P_{\lambda}^{(\alpha,\beta)}(x)$ as $P_{\lambda}^{(\alpha,-\alpha)}(x)$.

### 3. Derivation of the Representation Formula.

Again, throughout this section we shall assume $\alpha + \beta = 0$, $-\frac{1}{2} < \alpha$, $\beta < \frac{1}{2}$ and $\alpha \neq 0$. The case $\alpha = 0$ reduces to the representation of the Legendre functions and has been developed in [3].

The series representation that we will develop, in this section, for $P_{\lambda}^{(\alpha,-\alpha)}(x)$ is

$$P_{\lambda}^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin \pi \lambda}{\pi} \cdot$$

\begin{align*}
&\lambda(\lambda+1) \sum_{n=1}^{\infty} \frac{(2n+1)n!}{n(n+1)(\alpha+1)\Gamma(n+\lambda+n+1)} \left(\frac{1-x}{1+x}\right)^n, 0 \leq x < 1, \tag{3.1}
\end{align*}

and

\begin{align*}
P_{\lambda}^{(\alpha,-\alpha)}(x) &= \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin \pi \lambda}{\pi} \cdot \\
&\lambda(\lambda+1) \sum_{n=1}^{\infty} \frac{(2n+1)n!}{n(n+1)(\alpha+1)\Gamma(n+\lambda+n+1)} \left(\frac{1-x}{1+x}\right)^n + \frac{1}{\lambda(\lambda+1)} \left(\frac{1-x}{1-x}\right)^1, -1 < x \leq 0. \tag{3.2}
\end{align*}

In order to derive (3.1) and (3.2), we shall first introduce an auxiliary function $k(x;h)$, apply (2.7), (2.9) to $k(x;h)$ and utilize the uniqueness of the Jacobi transform.

**Lemma 3.1.** For $h \in (-1,1)$, define

$$k(x;h) = \begin{cases} 
\frac{1}{\alpha} \left[\frac{1+x}{1-x} - \left(\frac{1+x}{1-x}\right)^\alpha\right], & h \leq x < 1, \\
0, & -1 < x < h 
\end{cases}$$

Then
\begin{equation}
\mathring{k}(\alpha,-\alpha)(x;h)(\lambda) = \frac{1}{\lambda(\lambda+1)} \left( \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} - p_{\lambda}^{(\alpha,-\alpha)}(h) \right), \lambda \neq 0, \lambda \geq -\frac{1}{2}
\end{equation}

\begin{proof}
(2.2) together with (2.7) yields for \( \lambda \neq 0 \) and \( \alpha + \beta = 0 \)
\begin{align*}
\mathring{k}(\alpha,-\alpha)(\cdot;h)(\lambda) &= \frac{1}{2} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{-\alpha} p_{\lambda}^{(\alpha,-\alpha)}(x) k(x;h) \, dx \\
&= -\frac{1}{2} \frac{1}{\lambda(\lambda+1)} \int_{-1}^{1} \frac{d}{dx} ((1-x)^{\alpha+1}(1+x)^{-\alpha+1} \frac{d}{dx} p_{\lambda}^{(\alpha,-\alpha)}(x)) k(x;h) \, dx.
\end{align*}

On integrating by parts, we obtain
\begin{align*}
\mathring{k}(\alpha,-\alpha)(\cdot;h)(\lambda) &= \frac{1}{\lambda(\lambda+1)} \int_{h}^{1} \frac{d}{dx} p_{\lambda}^{(\alpha,-\alpha)}(x) \, dx \\
&\text{from which it follows that for } \lambda \neq 0, \lambda \geq -\frac{1}{2}
\end{align*}

\begin{equation}
\mathring{k}(\alpha,-\alpha)(\cdot;h)(\lambda) = \frac{1}{\lambda(\lambda+1)} \left( p_{\lambda}^{(\alpha,-\alpha)}(1) - p_{\lambda}^{(\alpha,-\alpha)}(h) \right)
\end{equation}

Equivalently,
\begin{equation}
\mathring{k}(\alpha,-\alpha)(\cdot;h)(\lambda) = \frac{1}{\lambda(\lambda+1)} \left( \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} - p_{\lambda}^{(\alpha,-\alpha)}(h) \right)
\end{equation}

from (2.3).

When \( \lambda = 0 \), \( p_{0}^{(\alpha,-\alpha)}(x) = 1 \). This together with (2.7) yields
\begin{align*}
\mathring{k}(\alpha,-\alpha)(\cdot;h)(0) &= \frac{1}{2} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{-\alpha} k(x;h) \, dx \\
&= \frac{1}{2} \int_{h}^{1} (1-x)^{\alpha}(1+x)^{-\alpha}\left( 1 - \frac{1}{a} \frac{(1+x)^{\alpha}}{1-x} \right) \, dx \\
&= \frac{1}{2} \frac{1}{h} \int_{h}^{1} \frac{1}{1-x} \, dx.
\end{align*}

This completes the proof of Lemma 3.1.

Since \( \lambda \mathring{k}(\alpha,-\alpha)(\cdot;h)(\lambda-\frac{1}{2}) \in C^{1}(R^{+}) \) and since \( k(x;h) \) is continuous on \((-1,1)\), it follows from (2.8) and Lemma 3.1 that for \( \lambda \neq 0 \)
\begin{align*}
k(x;h) &= 4 \int_{0}^{\infty} \frac{1}{\lambda^{2}-\frac{1}{4}} \left( \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda-\alpha+\frac{1}{2})\Gamma(\alpha+1)} - \frac{\Gamma^{2}(\lambda+\frac{1}{2}) p_{\lambda-\frac{1}{2}}^{(\alpha,-\alpha)}(h)}{\Gamma(\lambda+\alpha+\frac{1}{2})\Gamma(\lambda-\alpha+\frac{1}{2})} \right) \cdot p_{\lambda-\frac{1}{2}}^{(-\alpha,\alpha)}(-\chi) \sin \chi \, d\lambda.
\end{align*}

From (2.11) with \( \mu = 1, \sigma \geq 0, \beta(-1,1) \) and Lemma 3.1, we have
\[ \hat{k}(\alpha, -\alpha)(\cdot, h)(\sigma - h) = \frac{1}{\sigma^2 - h} \left[ \frac{\Gamma(\sigma + h - \alpha) \Gamma(\sigma - h)}{\Gamma(\sigma + h) \Gamma(\sigma - h)} - \hat{p}(\sigma, -\alpha)(h) \right] \]

\[ = \frac{\Gamma(\sigma + h)}{\pi(\sigma^2 - h)} \frac{k(\alpha, -\alpha)(\sigma, h)(0) \sin \pi(\sigma - h)}{\Gamma(\sigma + h) \Gamma(\sigma - h)} + \sum_{n=1}^{\infty} \frac{(2n+1) \Gamma(n+1) \Gamma(\sigma + h) \sin \pi(\sigma - n - h)}{\pi(\sigma^2 - (n+1)^2) \Gamma(n+1) \Gamma(\sigma + h) \Gamma(\sigma - n)} \frac{1}{n(n+1)}. \]

where \( \hat{k}(\alpha, -\alpha)(\cdot, h)(0) \) is as given in Lemma 3.1. Replacing \( \sigma \) by \( \lambda + \frac{1}{2} \) in the above expression together with Lemma 3.1 and the uniqueness of the Jacobi transform imply

\[ \frac{1}{\lambda(\lambda + 1)} \left[ \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1) \Gamma(\lambda)} - \frac{\Gamma(\lambda + \alpha + 1) \sin \pi \hat{k}(\alpha, -\alpha)(\lambda, h)(0)}{\pi(\lambda + 1) \Gamma(\lambda + 1) \Gamma(\lambda + 1)} \right] = \frac{\Gamma(\lambda + \alpha + 1) \sin \pi \hat{k}(\alpha, -\alpha)(\lambda, h)(0)}{\pi(\lambda + 1) \Gamma(\lambda + 1) \Gamma(\lambda + 1)} + \sum_{n=1}^{\infty} \frac{\Gamma(\lambda + n + 1)}{\pi(\lambda - n) \Gamma(n+1) \Gamma(\lambda + n+1) \Gamma(\lambda + 1) n(n+1)} \frac{1}{n(n+1)} \left[ \frac{\Gamma(\lambda + n + 1)}{\Gamma(\lambda + 1) n!} - \hat{p}(\alpha, -\alpha)(\lambda, h)(0) \right]. \]

Therefore,

\[ \hat{p}(\alpha, -\alpha)(\lambda, h)(0) = \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1) \Gamma(\lambda + 1)} \frac{\Gamma(\lambda + \alpha + 1) \sin \pi \hat{k}(\alpha, -\alpha)(\lambda, h)(0)}{\pi(\lambda + 1) \Gamma(\lambda + 1) \Gamma(\lambda + 1)} \]

\[ - \sum_{n=1}^{\infty} \frac{\lambda(\lambda + 1)(2n+1) \Gamma(\lambda + n+1) \sin \pi(\lambda - n)}{\pi(\lambda - n) \Gamma(n+1) \Gamma(\lambda + n+1) \Gamma(\lambda + 1)n(n+1)} + \sum_{n=1}^{\infty} \frac{\lambda(\lambda + 1)(2n+1)n! \Gamma(\lambda + n+1) \sin \pi(\lambda - n) \hat{p}(\alpha, -\alpha)(h)}{\Gamma(\lambda + 1) n(n+1) \Gamma(\lambda + n+1) n(n+1)} \]

(3.4)

From (2.7) we now have

\[ \hat{p}(\alpha, -\alpha)(0) = \frac{1}{2} \int_{-1}^{1} (1-x)\alpha(1+x)^{-\alpha} \hat{p}(\alpha, -\alpha)(x) \hat{p}(\alpha, -\alpha)(x) \, dx \]

which together with the above expression for \( \hat{p}(\alpha, -\alpha)(h) \) yields

\[ \hat{p}(\alpha, -\alpha)(0) = \frac{1}{2} \int_{-1}^{1} (1-x)\alpha(1+x)^{-\alpha} \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1) \Gamma(\lambda + 1)} \]

\[ - \frac{\Gamma(\lambda + \alpha + 1) \sin \pi \hat{k}(\alpha, -\alpha)(\lambda, h)(0)}{\pi(\lambda + 1) \Gamma(\lambda + 1) \Gamma(\lambda + 1)} \]

\[ - \sum_{n=1}^{\infty} \frac{(2n+1) \lambda(\lambda + 1) \Gamma(\lambda + n+1) \sin \pi(\lambda - n)}{\pi(\lambda - n) \Gamma(n+1) \Gamma(\lambda + n+1) \Gamma(\lambda + 1)n(n+1)} + \sum_{n=1}^{\infty} \frac{(2n+1) \lambda(\lambda + 1) \Gamma(\lambda + n+1) \sin \pi(\lambda - n) \hat{p}(\alpha, -\alpha)(x)}{\Gamma(\lambda + 1) n(n+1) \Gamma(\lambda + n+1) \Gamma(\lambda + 1) n(2n+1)} \, dx \]
Using Euler’s formula [5]
\[ \int_0^x (x-t)^\alpha t^\beta \, dt = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} x^{\alpha+\beta+1}, \quad \alpha, \beta > -1 \]  
(3.5)

with \( \alpha + \beta = 0 \), \( t = 1 + u \), we obtain
\[ \int_{-1}^1 (1-x)^\alpha (1+x)^{-\alpha} \, dx = 2\Gamma(\alpha+1)\Gamma(1-\alpha). \]

This together with Lemma 3.1 yields
\[ P_n(\alpha,-\alpha)(0) = \frac{\Gamma(1-\alpha)\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} \]

\[ - \sum_{n=1}^\infty \frac{\lambda(\lambda+1)(2n+1)\Gamma(\lambda-a)\Gamma(\lambda+a+1)\sin \pi (\lambda-n)}{n(n+1)\pi(\lambda-n)(\lambda+n+1)\Gamma(\lambda+1)} \int_{-1}^1 (1-x)^{\alpha}(1+x)^{-\alpha} \frac{1}{2\alpha} (1-x) \, dx \]

\[ + \frac{\Gamma(\lambda+a+1)\sin \pi \lambda}{2\pi \Gamma(a+1)\Gamma(\lambda+1)} \int_{-1}^1 (1-x)^{\alpha}(1+x)^{-\alpha} (1+x)^{\alpha} (1-x)^{-\alpha} \frac{1}{2\alpha} \int_{-1}^1 (1+t)^{\alpha} \, dt \, dx \]

\[ + \sum_{n=1}^\infty \frac{\lambda(\lambda+1)(2n+1)n!\Gamma(\lambda+a+1)\sin \pi (\lambda+n)}{\pi(\lambda-n)(\lambda+a+1)\Gamma(\lambda+1)\Gamma(n+1)n(n+1)} \cdot \]

\[ \cdot \int_{-1}^1 (1-x)^{\alpha}(1+x)^{-\alpha} \, \rho_\alpha^n(-\alpha)(x) \, dx. \]

The last term in the above expression vanishes by the orthogonality of the Jacobi polynomials; that is,
\[ \int_{-1}^1 (1-x)^{\alpha}(1+x)^{-\alpha} \, \rho_\alpha^n(-\alpha)(x) \, dx = \]

\[ = \int_{-1}^1 (1-x)^{\alpha}(1+x)^{-\alpha} \, \rho_\alpha^n(-\alpha)(x) \, \rho_\alpha^n(-\alpha)(x) \, dx = 0 \]

Moreover, using (3.5), the third term can be written
\[ \int_{-1}^1 (1-x)^{\alpha+1}(1+x)^{\alpha} \, dx = \Gamma(\alpha+2)\Gamma(1-\alpha). \]

Therefore,
\[ P_\lambda(\alpha,-\alpha)(0) = \frac{\Gamma(1-\alpha)\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} - \]

\[ - \sum_{n=1}^\infty \frac{\lambda(\lambda+1)(2n+1)\Gamma(\lambda-a)\Gamma(\lambda+a+1)\sin \pi (\lambda-n)}{n(n+1)\pi(\lambda-n)(\lambda+n+1)\Gamma(\lambda+1)} \]

\[ - \frac{\Gamma(\lambda+a+1)\sin \pi \lambda}{2\pi \Gamma(a+1)\Gamma(\lambda+1)} \Gamma(\alpha+2)\Gamma(1-\alpha) + \]

\[ + \frac{\Gamma(\lambda+a+1)\sin \pi \lambda}{4\pi \Gamma(\lambda+1)\Gamma(\alpha+1)\lambda} \int_{-1}^1 \int_{-1}^{1+t} (1+t)^{\alpha} \, dt \, dx. \]  
(3.6)
From (2.6), (2.7) and the identity $p_n^{(\alpha,\beta)}(-x) = (-1)^n p_n^{(\beta,\alpha)}(x)$, it follows that

$$\hat{p}_n^{(\alpha,\beta)}(0) = \frac{\Gamma(\lambda+\alpha+1)\Gamma(1-\alpha)\sin\pi\lambda}{\pi\lambda(\lambda+\lambda+1)}, \lambda > 0, \lambda > -\frac{1}{2} \quad (3.7)$$

Hence by the uniqueness of the Jacobi transform, we have from (3.6) and (3.7),

$$1 - \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)(2n+1)\sin\pi(\lambda-n)}{\pi\lambda(n+1)(\lambda+1)} \frac{a+1}{2a} \frac{\sin\pi\lambda}{\pi} +$$

$$+ \frac{\sin\pi\lambda}{4\pi\alpha\Gamma(1+\alpha)\Gamma(1-\alpha)} \left[ \int_{-1}^{1} \left( 1 - x^2 \right)^{\alpha-\frac{1}{2}} \frac{1}{1+t^2} dt \right] = \frac{\sin\pi\lambda}{\pi\lambda(\lambda+1)}$$

Now (3.4) can be expressed as

$$\frac{\Gamma(\lambda+1)\lambda\Gamma(\alpha+1)}{\Gamma(\lambda+\alpha+1)} p^{(\alpha,\alpha)}(x) =$$

$$= \lambda(\lambda+1) \sum_{n=1}^{\infty} \frac{(2n+1)n!\sin\pi(\lambda-n)p^{(\alpha,\alpha)}(x)}{\pi n(n+1)(\alpha+1)n\pi(\lambda-n)(\lambda+1)} +$$

$$+ \frac{\sin\pi\lambda}{\pi} \left[ \left( 1 + \frac{1}{\lambda(\lambda+1)} \right) \frac{1}{2a} \int_{-1}^{1} \left( 1 - x^2 \right)^{\alpha-\frac{1}{2}} \frac{1}{1+t^2} dt \right]$$

By interchanging the order of integration and by (3.5) we obtain

$$\int_{-1}^{1} \left( 1 - x^2 \right)^{\alpha-\frac{1}{2}} \frac{1}{1+t^2} dt = 2\Gamma(1+\alpha)\Gamma(2-\alpha)$$

Thus,

$$\frac{\Gamma(\lambda+1)\Gamma(\alpha+1)}{\Gamma(\lambda+\alpha+1)} p^{(\alpha,\alpha)}(x) =$$

$$= \lambda(\lambda+1) \sum_{n=1}^{\infty} \frac{(2n+1)n!(-1)^n\sin\pi\lambda p^{(\alpha,\alpha)}(x)}{\pi n(n+1)(\alpha+1)n\pi(\lambda-n)(\lambda+1)} +$$

$$+ \frac{\sin\pi\lambda}{\pi} \left[ \left( 1 + \frac{1}{\lambda(\lambda+1)} \right) \frac{1}{2a} + \frac{x}{2a} \right] \left( \int_{-1}^{1} \left( 1 - x^2 \right)^{\alpha-\frac{1}{2}} \frac{1}{1+t^2} dt \right)$$

$$\left( 1 + \frac{1}{\lambda(\lambda+1)} \right) \frac{1}{2a} + \frac{x}{2a} \int_{-1}^{1} \left( 1 - x^2 \right)^{\alpha-\frac{1}{2}} \frac{1}{1+t^2} dt \right] \quad (3.8)$$

The series representation of the Jacobi function $p^{(\alpha,\alpha)}(x)$ will be completed once we obtain an equivalent expression for the integral.

$$f(x;\alpha) = \int_{-1}^{1} \frac{1}{1+t^2} dt.$$

Lemma 3.2. For $-\frac{1}{2} < \alpha < \frac{1}{2}$, $(\alpha \neq 0)$, we have
a) \[ f(x;\alpha) = \frac{(1-x)^{\alpha+1}}{(1+x)^\alpha} \left( 1 - \frac{2\alpha}{1+x} \sum_{n=0}^{\infty} \frac{(-1)^n (1-x)^n}{a+n+1} \right), \quad 0 \leq x < 1 \]

b) \[ f(x;\alpha) = \frac{2\pi\alpha}{\sin\pi\alpha} \frac{(1-x)\alpha}{(1+x)\alpha-1} \left( 1 - \frac{2\alpha}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^n (1+x)^n}{a-n-1} \right), \quad -1 < x < 0 \]

**Proof:** a) Integration by parts yields the recursive relation

\[ f(x;\alpha + 1) = \frac{1}{\alpha + 1} \left( \frac{(1-x)^{\alpha+1}}{(1+x)^\alpha} - \frac{\alpha}{\alpha+1} f(x;\alpha + 1) \right). \]

By employing this relation and after simplification, we obtain

\[ f(x;\alpha) = \frac{(1-x)^{\alpha+1}}{(1+x)^\alpha} \left( 1 - \frac{2\alpha}{1+x} \sum_{n=0}^{\infty} \frac{(-1)^n (1-x)^n}{a+n+1} \right) \]

The series converges for all \( x \) such that \( \frac{1-x}{1+x} < 1 \); that is, if \( 0 < x < 1 \). When \( x = 0 \),

\[ f(0;\alpha) = 1 - 2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{a+n+1}. \]

b) We rewrite \( f(x;\alpha) \) as

\[ f(x;\alpha) = \int_0^1 (\frac{1-t}{1+t})^\alpha dt + \int_0^1 (\frac{1-t}{1+t})^\alpha dt = J(x;\alpha) + f(0,\alpha), \]

By introducing

\[ J^*(x;\alpha) = \int_{-1}^x (\frac{1-t}{1+t})^\alpha dt \]

\( J(x;\alpha) \) can be written as

\[ J(x;\alpha) = J^*(0;\alpha) - J^*(x;\alpha). \]

Upon an integration by parts, we obtain

\[ J^*(x;\alpha) = \frac{1}{1-\alpha} \frac{(1-x)\alpha}{(1+x)\alpha-1} + \frac{\alpha}{1-\alpha} J^*(x;\alpha-1) \]

Repeating the above formula, recursively, results in the series.

\[ J^*(x;\alpha) = \frac{(1-x)\alpha}{(1+x)\alpha-1} \left( 1 - \frac{2\alpha}{1-x} \sum_{n=0}^{\infty} \frac{(-1)^n (1+x)^n}{a-n-1} \right) \]

which converges for all \( x \) such that \( \frac{1+x}{1-x} < 1 \); that is, for \( -1 < x < 0 \).

When \( x = 0 \),

\[ J^*(0;\alpha) = 1 - 2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{a-n-1}. \]
Thus,

\[
f(x; a) = J^*(0, a) + f(0, a) - J^*(x; a) = \frac{2\pi a}{\sin \pi a} - \frac{(1-x)^a}{(1+x)^a} \left( 1 - 2a \sum_{n=0}^{\infty} \frac{(-1)^n}{a-n-1} \frac{(1+x)^n}{(1-x)^n} \right)
\]

which completes the verification of Lemma 3.2.

From (3.8) and Lemma 3.2, the representation of the Jacobi function \( p_{\lambda}^{(\alpha,-\alpha)}(x) \) will follow. In particular, for \( \lambda \geq -\frac{1}{2} \) (\( \lambda \neq 0 \))

\[
p_{\lambda}^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin \pi \lambda}{\pi} \left\{ \lambda(\lambda+1) \cdot \sum_{n=1}^{\infty} \frac{(2n+1)n!}{n(n+1)\Gamma(\lambda+\alpha+1)} \frac{(-1)^n}{n(n+1)} \frac{(1-x)^n}{(1+x)^n} \cdot \left[ 1 + \frac{1}{\lambda(\lambda+1)} \right] - \frac{(1-x)^n}{(1+x)^n} \right\}, 0 \leq x < 1;
\]

and

\[
p_{\lambda}^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin \pi \lambda}{\pi} \left\{ \lambda(\lambda+1) \cdot \sum_{n=1}^{\infty} \frac{(2n+1)n!}{n(n+1)\Gamma(\lambda+\alpha+1)} \frac{(-1)^n}{n(n+1)} \frac{(1-x)^n}{(1+x)^n} \cdot \left[ 1 + \frac{1}{\lambda(\lambda+1)} \right] - \frac{1}{\alpha} + \frac{(1+x)^\alpha}{(1-x)^\alpha} \frac{\pi}{\sin \pi \alpha} + \sum_{n=0}^{\infty} \frac{(-1)^n}{a-n-1} \frac{(1+x)^n}{(1-x)^n} \right\}, -1 < x < 0.
\]

The above representations will hold for \( \lambda = 0 \) provided that \( \frac{\sin \pi \lambda}{\pi \lambda} \) is interpreted to be equal to 1 for \( \lambda = 0 \). When \( \alpha = 0 \), the formula reduces to that for Legendre functions derived in [3], provided that \(- \frac{1}{\alpha} + \frac{(1+x)^\alpha}{(1-x)^\alpha} \frac{\pi}{\sin \pi \alpha} \) is given its limiting value of 0 as \( \alpha \to 0 \).

REFERENCES


