ON THE STRONG MATRIX SUMMABILITY
OF DERIVED FOURIER SERIES

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ABSTRACT. Strong summability with respect to a triangular matrix has been defined
and applied to derived Fourier series yielding a result which extends some known
results under a general criterion.

KEY WORDS AND PHRASES. Strong Summability, Toeplitz matrix, Fourier Series.

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1. INTRODUCTION.

The triangular matrix \( A = [a_{n,k}] \), \( n, k = 0,1,\ldots \) and \( a_{n,k} = 0 \) for \( k > n \) is
regular if

\[
\lim_{n \to \infty} a_{n,k} = 0, \quad n \in \mathbb{N}
\]

and

\[
\sum_{k=0}^{n} |a_{n,k}| \leq M, \quad M \text{ is independent of } n
\]

Denoting the sum \( \sum_{r=1}^{n} u_r \) by \( s_n \), Fekete [1], defined that the series \( \sum_{r=1}^{n} u_r \) is
strongly summable to the sum \( s \), provided

\[
\sum_{k=1}^{n} |s_k - s| = o(n).
\]

This type is now known as strong Cesàro summability of order unity with index 1 or
\([C,1]\) summability.

The series \( \sum_{r=1}^{n} u_r \) is said to be strongly summable by Cesàro means, with index \( q \),
or summable \([C,q]\), or summable \( H_q \) to the sum \( s \) if

\[
\sum_{k=1}^{n} |s_k - s|^q = o(n).
\]

A special point of interest in the method of summability \( H_q \) lies in the fact
that it is given neither by Toeplitz matrix nor by a sequence to function transforma-
tion. The relationship between summability $H_q$ and some regular methods of summation given by A-matrices has been investigated by Kuttner, [2], who proved that if A is any regular Toeplitz method of summability then for any $0 < q < 1$ there is a series which is not summable A but summable $H_q$.

In the present paper we shall define strong summability of series $\sum u_k$ with the help of a matrix.

**DEFINITION.** The series $\sum u_k$ is said to be strongly summable by the regular method A determined by the matrix $[a_{n,k}]$ with index $q (q > 0)$ to the sum $s$ if

$$\sum_{k=0}^{n} a_{n,k} s_k - s|^q = o(1), \quad n \to \infty.$$

For $a_{n,k} = \frac{1}{n+1}$, $k \leq n$, we get $(C,1)$ matrix.

2. **MAIN RESULTS.**

Let $f(x)$ be a periodic function with period $2\pi$ and integrable (L) over $(-\pi, \pi)$. Let

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of $f(x)$ and

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx)$$

be the first derived series of (2.1) obtained by term by term differentiation.

Write

$$g(u) = f(x+u) - f(x-u) - 2uf'(x), \quad (2.3)$$

where $f'(x)$ is the derivative of $f(x)$,

$$G(t) = \int_{0}^{t} |dg(u)|. \quad (2.4)$$

Here we shall take $q = 1, 2$. Since the case $q = 1$ is included in the strong summability for $q = 2$, we omit the same. Precisely we prove the following:

**THEOREM.** Let $g(u), G(t)$ be defined as in (2.3) and (2.4). If $g(u)$ is a continuous function of bounded variation over $[0, \pi]$ and for some $B \geq 1$

$$G(t) = o \left[ t \lambda^{\theta}(t) \right], \quad t \to 0, \quad (2.5)$$

where $\lambda^{\theta}(t)$ is a positive function of $t$ such that

$$\lambda^{\theta}(t) \to 0 \quad \text{as} \quad t \to 0, \quad (2.6)$$

it is monotonic in $(n^{-1}, \delta)$ ($\delta$ being small but fixed) and

$$\int_{n^{-1}}^{\delta} \frac{\lambda^{2\theta}(t)}{t} dt = O(1) \quad (2.7)$$

then the derived series (2.2) is strongly summable to $f'(x)$ by the matrix $(C,1)$ with index 2.

Note (2.7) is equivalent to $\frac{\lambda^{2}(t)}{t} \in L(0, \delta)$. 

In order to prove the theorem we need the following lemma.

**Lemma.** If \( G(t) = o(t) \) as \( t \to 0 \) then for small but fixed \( \delta \)

\[
\int_{n-1}^{\delta} \frac{|dg(t)|}{t} \, dt \int_{n-1}^{\delta} \frac{|dg(u)|}{u} \, du = o(n)
\]

and

\[
\int_{n-1}^{\delta} \frac{|dg(t)|}{t^2} \, dt \int_{n-1}^{\delta} \frac{|dg(u)|}{u} \, du = o(n) .
\]

**Proof.** Since

\[
\int_{n-1}^{\delta} \frac{|dg(u)|}{u} \, du = \left[ \frac{G(u)}{u} \right]_{n-1}^{\delta} + \int_{n-1}^{\delta} \frac{G(u)}{u^2} \, du
\]

\[= o(1) + \int_{n-1}^{\delta} o\left(\frac{1}{u}\right) \, du, \text{ in view of (2.4)}, \]

\[= o(\log n), \]

Therefore

\[
\int_{n-1}^{\delta} \frac{|dg(t)|}{t} \, dt \int_{n-1}^{\delta} \frac{|dg(u)|}{u} \, du = o(\log n)^2 = o(n).
\]

Again

\[
\int_{n-1}^{\delta} \frac{|dg(t)|}{t^2} \, dt \int_{n-1}^{\delta} \frac{|dg(u)|}{u} \, du
\]

\[= \int_{n-1}^{\delta} \frac{|dg(t)|}{t^2} \left\{ \left[ \frac{G(u)}{u} \right]_{n-1}^{t} + \int_{n-1}^{t} \frac{G(u)}{u^2} \, du \right\} \, dt
\]

\[= \int_{n-1}^{\delta} \frac{|dg(t)|}{t^2} \left\{ G(t) + o(1) + o(\log n t) \right\} \, dt
\]

\[= o(1) \left\{ \int_{n-1}^{\delta} \frac{dg(t)}{t^2} \log nt \right\}
\]

\[= o \left\{ \int_{n-1}^{\delta} \frac{G(t)}{t^2} \log nt \, dt - \int_{n-1}^{\delta} \frac{G(t)}{t^3} \, dt + 2 \int_{n-1}^{\delta} \frac{G(t)}{t^4} \log nt \, dt \right\}
\]

\[= o(n) + o \left( \int \frac{(1/u^2)}{1} \, du \right) + o \left[ \int (\log u/u^2) \, du \right]
\]

\[= o(n).
\]

3. **Proof of Theorem.**

The kth partial sum \( a_k(x) \) of the series (2.2) is given by [3],
Further, simplifying certain steps as given by [3] and [4] we have

\[ \sigma_k(x) - f'(x) = \frac{1}{2\pi} \int_{0}^{\pi} \sin \left(\frac{(k+1/2)t}{2}\right) \frac{dt}{\sin \frac{t}{2}}. \]

Therefore

\[ \int_{n-1}^{n} \sin \left(\frac{kt}{2}\right) \frac{dt}{t} = \frac{1}{\pi} \left( \int_{n-1}^{n} + \int_{n}^{n+1} \right) \sin \left(\frac{kt}{2}\right) \frac{dt}{t}. \]

On simplifying and using the first part of the lemma we obtain

\[ \sum_{k=1}^{n} \left( \sigma_k(x) - f'(x) \right)^2 = \frac{1}{2\pi^2} \int_{n-1}^{n} \left( \int_{n-1}^{n} \sin \left(\frac{nt}{2}\right) \frac{du}{u(u-t)} \right) \sin \left(\frac{nu-t}{2}\right) \frac{du}{u(u-t)} + o(n) \]

and

\[ \int_{n-1}^{n} \frac{\sin \left(\frac{nu-t}{2}\right)}{u(u-t)} \frac{du}{u} = \int_{n-1}^{n} \frac{\sin \left(\frac{nu-t}{2}\right)}{(u-t)} \frac{du}{u} \frac{du}{u-t}. \]

Therefore

\[ P_1 = \frac{1}{2\pi^2} \int_{n-1}^{n} \frac{\sin \left(\frac{nu-t}{2}\right)}{u(u-t)} \frac{du}{u} + \frac{1}{2\pi^2} \int_{n-1}^{n} \frac{\sin \left(\frac{nu-t}{2}\right)}{u(u-t)} \frac{du}{u}. \]
\[ \begin{align*}
&= \frac{1}{\pi} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{1}{t^2} \int_{n-1}^{t} \sin n(u-t) \, dg(u) \\
&= \frac{1}{\pi} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{1}{t^2} \int_{n-1}^{t} \frac{\sin n(u-t)}{u-u(t)} \, dg(u) + o \left( \int_{n-1}^{t} \frac{|dg(t)|}{u} \int_{n-1}^{t} \frac{|dg(u)|}{u} \right) \\
&= \frac{1}{\pi} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{1}{t^2} \int_{n-1}^{t} \frac{\sin n(u-t)}{(u-t)} \, dg(u) + o(n)
\end{align*} \]

by virtue of the second part of the lemma.

Similarly it can be proved that \( P_2 = o(n) \). Thus we get

\[ \sum_{k=1}^{n} (a_k(x) - f'(x))^2 = \frac{1}{\pi} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{1}{t^2} \int_{n-1}^{t} \frac{\sin n(u-t)}{u-u(t)} \, dg(u) + o(n). \]

Integration by parts gives

\[ \int_{n-1}^{t} \frac{dg(u)}{(u-t)} \sin n(u-t) = \left[ \frac{\sin n(u-t)}{(u-t)} \int_{n-1}^{t} \frac{dg(u)}{(u-t)} \right]_{n-1}^{t} - \int_{n-1}^{t} \left[ \frac{n \cos n(u-t)}{(u-t)} - \frac{\sin n(u-t)}{(u-t)^2} \right] \, dg(u) \, du. \]

Using (2.5) this is equal to

\[ \left[ \frac{\sin n(u-t)}{(u-t)} \log \frac{t}{n} \lambda^\beta(t) \right]_{n-1}^{t} - o \left( \int_{n-1}^{t} t \log \frac{t}{n} \lambda^\beta(t) \frac{\cos n(u-t)}{(u-t)} \, du \right) \\
+ o \left( \int_{n-1}^{t} \frac{\sin n(u-t)}{(u-t)^2} \left( t \lambda^\beta(t) \right) \, du \right) = o \left( nt \lambda^\beta(t) \right). \]

Therefore

\[ \sum_{k=1}^{n} (a_k(x) - f'(x))^2 = o(n) \left[ \int_{n-1}^{t} \frac{dg(t)}{t} \lambda^\beta(g) \right] + o(n) \]

\[ = o(n) \left[ G(t) \lambda^\beta(t) \right]_{n-1}^{t} + o(n) \left[ \int_{n-1}^{t} \frac{dg(t)}{t^2} \lambda^\beta(t) \, dt \right] \\
+ o(n) \left[ \int_{n-1}^{t} \frac{G(t)}{t} \left\{ \beta \lambda^{\beta-1}(t) \lambda'(t) \right\} \, dt \right] \\
= o(n) + o(n) \left[ \int_{n-1}^{t} \frac{\lambda^{2\beta}(t)}{t} \, dt \right] \\
+ o(n) \left[ \int_{n-1}^{t} \beta \lambda^\beta(t) \lambda^{\beta-1}(t) \lambda'(t) \, dt \right]. \]
\[ d \sum_{n} \frac{1}{n} \frac{d}{dt} \{ \lambda^{2\theta}(t) \} \, dt \]

\[ = o(n) \text{ by the hypothesis (2.7).} \]

Since \( \lambda^{\theta}(t) \) is monotonic, hence its differential coefficient is of constant sign. Thus we get

\[ n \sum_{k=1}^{n} |a_{n,k} - f'(x)|^2 = o(n) \]

and therefore

\[ n \sum_{k=1}^{n} a_{n,k} |o_{k}(x) - f'(x)|^2 = o(n) \]

This completes the proof of the theorem.

4. SPECIAL CASES.

By way of an application of our theorem, we take \( \theta = 1, \lambda(t) = 1/\log (1/t) \) and \( a_{n,k} = 1 \) then the following result follows, [4]:

THEOREM (Sharma). At a point for which \( f'(x) \) exists and

\[ G(t) = o\left( t/\log \frac{1}{t} \right) \text{ as } t \to 0 \]

then

\[ n \sum_{k=1}^{n} |o_{k}(x) - f'(x)|^2 = o(n \log \log n) \]

Since the above theorem is an extension of the result from [C, 1] summability to the case of [C, 2] summability, (Prasad and Singh [3]), our theorem further extends that result under a general type of criterion.

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