ABSTRACT. Let R be a ring (not necessarily with identity) and let E denote the set of idempotents of R. We prove that R is a direct sum of a J-ring (every element is a power of itself) and a nil ring if and only if R is strongly \(\mathfrak{m}\)-regular and E is contained in some J-ideal of R. As a direct consequence of this result, the main theorem of [1] follows.

KEY WORDS AND PHRASES. Periodic, potent, J-ring, nil ring, strongly \(\mathfrak{m}\)-regular ring, direct sum.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 16A15; Secondary 16A70.

1. INTRODUCTION.

Throughout the present note, R will represent a ring (not necessarily with identity), N the set of nilpotent elements of R, and E the set of idempotents of R. We say that R is periodic if for each \(r \in R\), there exist distinct positive integers \(h, k\) for which \(r^h = r^k\). According to Chacron's theorem (see, e.g., [2, Theorem 1]), R is periodic if and only if for each \(r \in R\), there exists a polynomial \(f(\lambda)\) with integer coefficients such that \(r - r^2f(r) \in N\). An element \(r\) of R is called potent if there is an integer \(n > 1\) such that \(r^n = r\). We denote by I the set of potent elements of R. If R coincides with I, R is called a J-ring. As is well known, every J-ring is commutative (Jacobson's theorem). An ideal of R is called a J-ideal if it is a J-ring. Also, we denote by \(I_0\) the set \(\{r \in R \mid r \text{ generates a subring with identity}\}\). It is clear that \(E \subseteq I \subseteq I_0\). Furthermore, if \(I_0\) is a subring of R then \(I_0\) coincides with I. In fact, if \(r\) is an arbitrary element of \(I_0\) then there exists a polynomial \(f(\lambda)\) with integer coefficients such that \(r = r^2f(r)\). This proves that \(I_0\) is a reduced periodic ring, and therefore a J-ring. Especially, R is a J-ring if and only if \(R = I_0\). If R is the direct sum of a J-ideal \(I'\) and a nil ideal \(N'\), then it is easy to see that \(I' = I = I_0\) and \(N' = N\).

2. MAIN THEOREM.

Now, the main theorem of this note is stated as follows:
THEOREM 1. The following conditions are equivalent:

1. \( R \) is right (or left) \( \pi \)-regular and \( E \) is contained in some \( J \)-ideal \( A \) of \( R \).
2. \( R \) is periodic and \( E \) is contained in some reduced ideal \( A \) of \( R \).
3. \( R \) is a direct sum of a \( J \)-ring and a nil ring.

More precisely, if 1) or 2) is satisfied, then \( N \) is an ideal of \( R \), \( R = A \oplus N \), and \( A = I = I_0 \). In particular, if \( R \) is right (or left) \( s \)-unital, that is, \( r \epsilon rR \) (or \( r \epsilon Rr \)) for all \( r \epsilon R \), then each of 1), 2) is equivalent to
4. \( R \) is a \( J \)-ring.

PROOF. Obviously, 3) \( \Rightarrow \) 2) \( \Rightarrow \) 1).

1) \( \Rightarrow \) 3). By a result of Dischinger (see, e.g., [3, Proposition 2]), \( R \) is strongly \( \pi \)-regular. Let \( r \) be an arbitrary element of \( R \). Then there exists a positive integer \( n \) and elements \( s', s'' \) of \( R \) such that \( r^{2n}s' = s'r^{2n} = r^n \). We put \( s = r^n s'^2 \). As is easily seen,

\[ s = s'' r^n s' = s'' r^n \]

and

\[ r^n s' r^n = s'' r^{2n} = r^n = r^{2n}s' = r^n s'' r^n. \]

Hence,

\[ r^n s = r^n s'' r^n s' = s'' r^{2n} = r^n = s'' r^n s'' r^n = s'' r^n \]

and

\[ r^{2n}s = r^n s'r^n = r^n s'r^n = r^n. \]

Since \( e = r^n s \) is an idempotent with \( re = er ( \epsilon A) \) and \( r^n e = r^n \), we see that

\[ (r - re)^n = r^n (1 - e)^n = 0. \]

This together with \( r = re + (r - re) \) proves that \( r \) is represented as a sum of an element in \( A \) and a nilpotent element. Now, let \( a, b \epsilon A \), and \( x, y \epsilon N \). Noting that \( xa.yb = xyba \) and \( ax.by = baxy \), we can easily see that \( xa \epsilon N \cap A = 0 \) and \( ax = 0; NA = AN = 0 \). Set \( xy = c + u \) and \( x+y = d + v \) (\( c, d \epsilon A \) and \( u, v \epsilon N \)), where we may assume that \( u^\ell = v^\ell = 0 \). In view of \( NA = 0 \), we obtain

\[ (xy)^2 = xy(c + u) = xyu \]

and

\[ (x+y)^2 = (x+y)(d+v) = (x+y)v, \]

and therefore

\[ (xy)^{\ell+1} = xyu^\ell = 0 \]

and

\[ (x+y)^{\ell+1} = (x+y)v^\ell = 0. \]

We have thus seen that \( N \) forms an ideal of \( R \) and \( R = A \oplus N \).
Given an integer \( n > 1 \), we denote by \( I_n \) the set \( \{ r \in R \mid r^n = r \} \). In [1], Abu-Khuzam and Yaqub proved that if \( R \) is a periodic ring with \( N \) commutative and for which \( I_n \) forms an ideal, then \( R \) is a subdirect sum of finite fields of at most \( n \) elements and a nil commutative ring. The next corollary includes this result.

**COROLLARY 1.** If \( R \) is periodic and \( I_n \) forms an ideal of \( R \) for some integer \( n > 1 \) then \( R = I_n \oplus N \) and \( I \) is a subdirect sum of finite fields of at most \( n \) elements.

**REFERENCES**
