ABSTRACT. In [1] Laurent Schwartz introduced the spaces $\mathfrak{O}_M$ and $\mathfrak{O}_C$ of multiplication and convolution operators on temperate distributions. Then in [2] Alexandre Grothendieck used tensor products to prove that both $\mathfrak{O}_M$ and $\mathfrak{O}_C$ are bornological. Our proof of this property is more constructive and based on duality.

KEY WORDS AND PHRASES. Temperate distribution, multiplication and convolution, inductive and projective limit, bornological, reflexive, and Schwartz spaces.

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We use $\mathbb{C}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{Z}$, resp., for the set of all complex, nonnegative integer, real, and integer numbers. For each $q \in \mathbb{N}$, the space

$$L_q = \{ f: \mathbb{R}^n \to \mathbb{C}; \| f \|^2 = \sum_{|\alpha| \leq q} \int_{\mathbb{R}^n} x^{2|\alpha|} |D^{\alpha} f(x)|^2 \, dx < +\infty \}$$

is Hilbert. Here $D^\alpha f$ stands for the Sobolev generalized derivative. We denote by $L_{-q}$ the strong dual of $L_q$ and by $\| \cdot \|_q$ the standard norm on $L_{-q}$. Then the space $\mathcal{S}$ of rapidly decreasing functions, resp. its strong dual $\mathcal{S}'$, is the $\text{proj lim } L_q$, resp. $\text{ind lim } L_{-q}$.

It is convenient to introduce the weight-function $W(x) = (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^n$. The mapping $T_k: f \mapsto w^k f: \mathcal{S}' \to \mathcal{S}'$, $k \in \mathbb{Z}$, is injective. We denote by $w^k L_m$, $k, m \in \mathbb{Z}$, the image of $L_m$ under $T_k$ and provide it with the topology which makes $T_k: L_m \to w^k L_m$ a topological isomorphism. Further, $\mathfrak{O}_q$, $q \in \mathbb{N}$, stands for the $\text{ind lim } w^p L_q$, and $\mathfrak{O}_{-q}$ for its strong dual. It is proved in [7] that for each $q \in \mathbb{N}$, the space $\mathfrak{O}_q$ is reflexive and $\mathfrak{O}_{-q} = \text{proj lim } w^p L_{-q}$. Finally, the space $\mathfrak{O}_M$ of multiplication operators on $\mathcal{S}'$ equals $\text{proj lim } \mathfrak{O}_q$, see [6].

PROPOSITION 1. The strong dual $\mathfrak{O}_M'$ of $\mathfrak{O}_M$ equals $\text{ind lim } \mathfrak{O}_{-q}$.

PROOF. The space $\mathcal{S}$ is dense in each $L_q$, $q \in \mathbb{N}$. Hence $\mathcal{S} = \text{ind lim } w^p \mathcal{S}$ is dense in $w^p L_q$ for each $p \in \mathbb{N}$. Then $\mathcal{S}$, and a fortiori its superset $\mathfrak{O}_M'$, are dense in each
\(\mathcal{O}_q = \text{ind lim } W^p L_q, q \in \mathbb{N}\). By [3, ch. IV, 4.4], the dual of \(\mathcal{O}_M\), equipped with the Mackey topology, equals \(\text{ind lim } \mathcal{O}_q\). The Mackey and strong topologies on \(\mathcal{O}_M\) coincide since \(\mathcal{O}_M^\prime\), as a projective limit of reflexive spaces \(\mathcal{O}_q\), is semireflexive, see [3, ch. IV, 5.5].

**PROPOSITION 2.** \(\mathcal{O}_M\) is the strong dual of \(\text{ind lim } \mathcal{O}_q\).

**PROOF.** By [3, ch. IV, 4.5], the topology \(\tau\) of \(\mathcal{O}_M = \text{proj lim } \mathcal{O}_q\) is consistent with the duality \(\mathcal{O}_M^\prime, \mathcal{O}_M\). Hence \(\tau\) is coarser than the strong topology \(\beta(\mathcal{O}_M^\prime, \mathcal{O}_M^\prime)\). On the other hand, it is proved in [5, Prop. 4] that \(\tau\) is finer than \(\beta(\mathcal{O}_M^\prime, \mathcal{O}_M^\prime)\).

**THEOREM 1.** The space \(\mathcal{O}_M\) is reflexive and \(\mathcal{O}_M^\prime\) is the strong dual.

**LEMMA 1.** Let \(r = 1 + \{1/n\}, q \in \mathbb{N}\). Then \(W^r L_q \subset L_q\) and every set bounded in \(W^r L_q\) is relatively compact in \(L_q\).

**PROOF.** Let \(B\) be an absolutely convex, bounded, and closed, set in \(W^r L_q\). Then \(B\) is weakly compact as a polar of a neighborhood in \(W^r L_q\). By [3, Ch. IV, 11.1, Cor 2], \(B\) is weakly sequentially compact and every sequence in \(B\) contains a subsequence \(\{f_k\}\) which converges weakly to some \(g \in B\). We may assume \(g = 0\).

Since the set \((W^r f; f \in B)\) is bounded in \(L^2(\mathbb{R}^n)\), the set \((W^q f; f \in B)\) is bounded in \(L^1(\mathbb{R}^n)\) and for any \(\alpha \in \mathbb{N}^n, |\alpha| \leq q\), the set \((\partial^\alpha f; f \in B)\), where \(\partial^\alpha f\) is the Fourier transform of \(f\), is uniformly bounded and locally equicontinuous on \(\mathbb{R}^n\). Hence \(\{f_k\}\) contains a subsequence, let it be again \(\{f_k\}\), such that \((\partial^\alpha f_k(x))\) converges uniformly to \(0\) on \(\mathbb{R}^n\) for all \(\alpha \in \mathbb{N}^n, |\alpha| \leq q\).

**LEMMA 2.** Let \(r = 1 + \{1/n\}, q \in \mathbb{N}\). Then \(W^r L_q \subset L_q\) and every set bounded in \(W^r L_q\) is relatively compact in \(L_q\).

**PROOF.** Let \(B\) be an absolutely convex, bounded, and closed, set in \(W^r L_q\). By the same argument as in Lemma 1, every sequence in \(B\) has a subsequence \(\{f_k\}\) which converges weakly to some \(g \in B\). We again assume \(g = 0\).

Denote by \(\|\cdot\|_{-r,q}\), resp. \(\|\cdot\|_{r,q}\), the norm in \(W^r L_q\), resp. \(W^r L_q\). Let \(A\) be the closed unit ball in \(L_q\), \(B_0\) the open unit ball in \(W^r L_q\), and \(a = \sup\{\|f\|_{-r,q}; f \in B\}\). Choose \(\epsilon > 0\). By Lemma 1, \(A\) is compact in the topology of \(W^r L_q\). Since \(L_q\) is dense in \(W^r L_q\), there exists a finite set \(\{\varphi_i; i \in F\} \subset L_q\) such that \(A \subset \bigcup\{\varphi_i + \epsilon B_0; i \in F\}\). For any \(f \in A\), there exists \(\varphi_i\) such that \(\|f - \varphi_i\|_{r,q} < \epsilon\) and for any \(k \in \mathbb{N}\) we have

\[|\langle \varphi_i f_k > \| \leq |\langle \varphi_i > | + |\langle \varphi_i, f_k > | \leq \|\varphi_i\|_{r,q} \cdot \|f_k\|_{-r,q} + |\langle \varphi_i, f_k > | \leq \|\varphi_i\|_{r,q} \cdot \|f_k\|_{-r,q} + \epsilon a + |\langle \varphi_i > | \leq \epsilon a + |\langle \varphi_i > |.\]

If we choose \(k_0 \in \mathbb{N}\) so that \(|\langle \varphi_i, f_k > | < \epsilon\) for all \(i \in F\) and \(k > k_0\) and the sequence \(\{f_k\}\) converges in \(L_q\).
PROPOSITION 3. For each $q \in \mathbb{N}$, $\mathcal{B}_q$ is a Schwartz space.

PROOF. By Lemma 2, for every $p \in \mathbb{N}$ the closed unit ball is $\mathcal{W}^{-r,p}_L$, where $r = 1 + \lceil \frac{1}{2n} \rceil$, is compact in $\mathcal{W}^{-p}_L$. By [4, Ch. 3.15, Prop. 9], the space $\mathcal{B}_q = \text{proj lim}_{p \to \infty} \mathcal{W}^{-p}_L$ is Schwartz.

PROPOSITION 4. Let $E_1 \subset E_2 \subset \ldots$ be locally convex spaces with identity maps: $E_k \to E_{k+1}$, $k \in \mathbb{N}$, continuous and $E = \text{ind lim}_{k \to \infty} E_k$ Hausdorff. Assume:

1. every set bounded in $E$ is bounded in some $E_k$,
2. every $E_k$ is a Schwartz space.

Then $E$ is a Schwartz space.

Proposition 4 is slightly more general than Prop. 8 in [4, Ch. 3.15] and its proof requires only minor changes of the proof presented in [4].

THEOREM 2. $\mathcal{B}$ is a Schwartz space.

PROOF. We have $\mathcal{B}' = \text{ind lim}_{q \to \infty} \mathcal{B}_q$. Each space $\mathcal{B}_q$ is Schwartz and Fréchet. Further, $\mathcal{B}'$ is reflexive, hence quasi-complete, which in turn implies fast completeness. By [8, Th. 1], the assumption (1) of Prop. 4 is satisfied and $\mathcal{B}$ is a Schwartz space.

THEOREM 3. $\mathcal{B}'$ is complete.

PROOF. The space $\mathcal{B}$ of $C^\infty$-functions, whose derivatives vanish at $\infty$ was introduced in [1]. We denote the space $\mathcal{B}^m$ by $\mathcal{B}_m$ and provide it with the topology for which $f \to \mathcal{W}^m f : \mathcal{B} \to \mathcal{B}_m$ is a topological isomorphism. Then the strong dual $\mathcal{B}'_C$ of $\mathcal{B}'$ equals $\text{ind lim}_{m \to \infty} \mathcal{B}_m$, see [2, Ch. 2, 4.4]. Also, $\mathcal{B}'_C$ is isomorphic to $\mathcal{B}'_M$ via Fourier transformation. Hence it suffices to prove that $\text{ind lim}_{m \to \infty} \mathcal{B}_m$ is complete.

Let $F$ be a Cauchy filter on $\mathcal{B}_C$, $G$ a filter of all $0$-neighborhoods in $\mathcal{B}_C$, and $H$ the filter with base \{(A + B; A \in F, B \in G). By [4, Ch. 2.12, Lemma 3], there exists $m \in \mathbb{N}$ such that $H$ induces a filter $H_m$ on $\mathcal{B}_m$ which is Cauchy in the topology inherited from $\mathcal{B}_C$. On each ball $\{x \in \mathbb{R}^n, |x| \leq n\}$, $r > 0$, the filter $H_m$ converges uniformly pointwise to a function $f \in \mathcal{B}$. Then $f$ adheres to $H_m$ on the subset $\mathcal{B}_m$ of $\mathcal{B}_C$ and by [4, Ch. 2.9, Prop. 1] the filter $F$ converges to $f$.

THEOREM 4. The spaces $\mathcal{B}'_N$ and $\mathcal{B}_M'$ are ultrabornological.

PROOF. By Exercise 9 in [4, Ch. 3.15], the strong dual of a complete Schwartz space is ultrabornological. Hence $\mathcal{B}_N$ is ultrabornological by Theorems 1, 2, and 3.

The space $\mathcal{B}_M'$ is ultrabornological as an inductive limit of Fréchet spaces $\mathcal{B}_q'$, $q \in N$.

THEOREM 5. The spaces $\mathcal{B}_C$ and its strong dual $\mathcal{B}'_C$ are both complete, reflexive, and ultrabornological spaces.

PROOF. The space $\mathcal{B}_C$ is complete as a strong dual of a bornological space. Since the Fourier transformations $F : \mathcal{B}_M \to \mathcal{B}'_C$ and $F : \mathcal{B}'_M \to \mathcal{B}_C$ are topological isomorphisms. Theorem 5 follows from Theorems 1, 3, and 4.

REFERENCES


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