AN EXTENSION OF A RESULT OF CSISZAR

P. B. CERRITO

Department of Mathematics
University of South Florida
Tampa, Florida 33620

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ABSTRACT. We extend the results of Csiszar (Z. Wahr. 5(1966) 279-295) to a topological semigroup $S$. Let $\mu$ be a measure defined on $S$. We consider the value of $\alpha = \sup \limsup_{n \to \infty} \mu^n(Kx^{-1})$. First, we show that the value of $\alpha$ is either zero or one. If $\alpha = 1$, we show that there exists a sequence of elements $(a_n)$ in $S$ such that $\mu^n = \delta_{a_n}$ converges vaguely to a probability measure where $\delta$ denotes point mass. In particular, we apply the results to inverse and matrix semigroups.

KEY WORDS AND PHRASES. Topological semigroup, Infinite convolutions.

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1. INTRODUCTION.

Csiszar [1] proved the following result concerning a regular probability measure $\mu$ on a locally compact, second countable, Hausdorff group $G$: Either $\sup \mu^n(Kx^{-1}) \to 0$ as $n \to \infty$ for all compact sets $K$, where $\mu^n$ denotes the $n$-fold convolution of $\mu$, or there exists a sequence of elements $(a_n)$ such that $\mu^n = \delta_{a_n}$ converges vaguely to a probability measure where $\delta_{a_n}$ denotes point mass at $a_n$.

We will extend this result to probability measures defined on certain types of locally compact, second countable, Hausdorff semigroups which satisfy condition (c): If $A$ and $B$ are compact then so are $AB^{-1}$ and $A^{-1}B$ where

$$AB^{-1} = \{y: \text{there exists } z \in B \text{ such that } yz \in A\}.$$
We will also consider $\mu^n(Kx^{-1})$ when $\mu$ is defined on a semigroup $S$ of $m \times m$ matrices. A matrix semigroup does not necessarily satisfy condition (c).

To each regular probability measure $\mu$ on a semigroup $S$ we associate the value $\alpha_0 = \sup_{k \leq n} \limsup_{n \to \infty} \sup_{x \in S} \mu^n(Kx^{-1})$. We first show that $\alpha_0 = 0$ or $\alpha_0 = 1$. If $\alpha_0 = 0$ then $\mu^n Kx^{-1} \to 0$ vaguely for any sequence of elements $(a_n)$ in $S$. If $\alpha_0 = 1$ we find $(a_n)$ such that $\mu^n Kx^{-1} \to Kx^{-1}$ converges to a probability measure.

2. PRELIMINARY RESULTS.

In order to show the main results, we need the following lemma. We omit its proof since it is quite similar to an argument of Csiszar [1].

**Lemmas 1.** Assume $S$ satisfies condition (c). Let $\mu_1$ be a probability measure such that $\sup \mu_1(Kx^{-1}) \leq a$ for a compact set $K \subset S$. Then there exists a compact set $K_2$ (depending on $\mu_1$) such that for any other probability measure on $S$,

$$\mu_1 \ast \mu_2(Kx^{-1}) \leq a - a/2(1 - \mu_2(K_2x^{-1})).$$

Define $\alpha_n(K) = \sup_x \mu^n(Kx^{-1})$. Then if $K < n$,

$$\mu^n(Kx^{-1}) = \int \mu^k(Kx^{-1}y^{-1}) \mu^n(dy) \leq \alpha_k(k) \int \mu^n(dy) = \alpha_k(K).$$

Therefore $(\alpha_n(K))$ is a nonincreasing sequence. Define $a(K) = \lim_{n \to \infty} \alpha_n(K)$ and $\alpha_0 = \sup_{K \text{compact}} a(K)$.

**Theorem 1.** If $S$ satisfies condition (c) then either $\alpha_0 = 0$ or $\alpha_0 = 1$.

**Proof.** Suppose $0 < \alpha_0 < 1$. Then there exists an $a$ such that $0 < a/2 < \alpha_0 < a < 1$. For any compact set $K$, there exists a $k(K)$ such that $\sup \mu^k(Kx^{-1}) < a$. Applying Lemma 1 to $\mu_1 = \mu^k$ and $\mu_2 = \mu^n$, yields the fact that for some $K_2$,

$$\mu^n(Kx^{-1}) \leq a - a/2(1 - \mu^n(K_2x^{-1})).$$

If $n$ is sufficiently large, $\mu^n(K_2x^{-1}) < a$ for all $x$ since

$$\sup \mu^n(K_2x^{-1}) < a(K_2) \leq a_0 < a.$$ 

But then $\mu^n(Kx^{-1}) \leq a - a/2(1 - a) = a(a+1)/2.$
Therefore $\alpha(K) \leq \alpha(1+\alpha)/2$. Since $K$ is arbitrary we have a contradiction. We conclude that $\alpha_0 = 0$ or $\alpha_0 = 1$.

QED

Before proceeding we present an example. Let $S = [0, \infty)$ with the usual topology. Define multiplication by $r \cdot s = \max(r, s)$. Let $K = [0, n]$ be a compact subset of $S$. Then

$$K^{x-1} = \begin{cases} 0 & x > n \\ K & x \leq n \end{cases}$$

and

$$\mu^n(K^{x-1}) = \begin{cases} 0 & x > n \\ \mu(K) & x \leq n. \end{cases}$$

Therefore if $\mu$ has compact support then $\alpha_0 = 1$. Otherwise, $\alpha_0 = 0$.

3. MATRIX SEMIGROUPS

Let $S$ be the set of all $m \times m$ matrices with probability measure $\mu$ defined on $S$ such that the support of $\mu$ generates a subsemigroup $S_\mu$ of $S$. We assume $S$ has the usual topology. Define $G = \{X \in S : X$ is nonsingular$\}$. Then $G$ forms a subgroup of $S$. We want to consider the subgroup $G_\mu$ of $G$ generated by the set $S_\mu \cap G$. We consider the case where $G_\mu$ is locally compact. Then $G_\mu$ becomes a topological subgroup of $S$. If $\mu(G) = 1$ then we need only apply Csiszar [1] to show that $\alpha_0 = 0$ or $\alpha_0 = 1$. Therefore we assume $0 < \mu(G) < 1$. Define a measure $\mu'$ on $G$ such that

$$\mu'(B) = \frac{\mu(B \cap G)}{\mu(G)}$$

for $B \in S$.

Then $(\mu')^2(B) = \int_S \mu'(Bx^{-1}) \mu'(dx)$

$$= \int_G \mu(Bx^{-1} \cap G)/\mu(G) \mu'(dx)$$

$$= 1/\mu(G) \int_G \mu(Bx^{-1} \cap G)/\mu(G) \mu(dx)$$

Now $Bx^{-1} \cap G = \{y \in G : yx \in B\} = \{y \in S : yx \in B \cap G\}$ if $x \in G$. Therefore,

$$(\mu')^2(B) = 1/\mu(G)^2 \int_G \mu((B \cap G)x^{-1}) \mu(dx).$$

If $x \in G$ then $(B \cap G)x^{-1} = 0$. Therefore

$$(\mu')^2(B) = 1/\mu(G)^2 \int_S \mu((B \cap G)x^{-1}) \mu(dx)$$

$$= \mu^2(B \cap G)/\mu(G)^2.$$ 

By an induction argument,

$$(\mu')^n(B) = \mu^n(B \cap G)/\mu(G)^n.$$
Define the following notation:

\[
\alpha_g = \sup_{K \subset G} \lim_{n} \sup_{x \in S} (\mu')^n(Kx^{-1})
\]

\[
= \sup_{K \subset G} \lim_{n} \sup_{x \in S} (\mu')^n(Kx^{-1})
\]

\[
= \sup_{K \subset G} \lim_{n} [\sup_{x \in S} (\mu)^n(Kx^{-1})]/(\mu(G))^n.
\]

Since \(\mu(G) < 1\), \(\mu(G)^n \to 0\) as \(n \to \infty\). By Csiszar's result [1] for groups, either \(\alpha_g = 0\) or \(\alpha_g = 1\). However,

\[
\lim [\sup_{K} (\mu)^n(Kx^{-1})]/(\mu(G))^n < \infty.
\]

This is only possible if \(\lim \sup_{K} (\mu)^n(Kx^{-1}) = 0\) for any \(K \subset G\). Henceforth, we assume that \(K\) is a compact set consisting of singular matrices. We will also exclude the zero matrix from our discussion since \(0^{-1}=0\) reduces the problem to a triviality and it is obvious that \(\alpha_0 = 1\). That is, we define

\[
\alpha_0 = \sup_{K \subset S} \lim_{n} \sup_{x \neq 0} (\mu)^n(Kx^{-1}).
\]

We give an example. Suppose \(S\) consists of matrices with nonnegative entries such that for any \(X \in S\), every entry in \(X\) is contained in the set \([\delta, \infty)\)

where \(\delta > 1/m\). Then

\[
\mu^{n+1}(Kx^{-1}) = \int \cdots \int (K(y_n \cdots y_1 x)^{-1}) \mu(dy_1) \cdots \mu(dy_n)
\]

where \(K(y_n \cdots y_1 x)^{-1} = (z \in S: z y_n \cdots y_1 x \in K)\) and

\[
y_n \cdots y_1 x = \begin{bmatrix}
w_{11} & \cdots & w_{1m} \\
\vdots & & \vdots \\
w_{m1} & \cdots & w_{mm}
\end{bmatrix}
\]

where \(w_{ij}\) has minimal value \(m_i^{-1}e^{-n}\) for all \(i\) and \(j\). Therefore for

\[Z = (z_{ij}) \in K(y_n \cdots y_1 x)^{-1} \cdot m_i^{-1}n \sum z_{ij} \in K\]

so that as \(n \to \infty\), \(\sum z_{ij} \to 0\) for all \(i\). Hence for any compact set \(K\),

\[
\lim_{n} \mu^n(Kx^{-1}) = 0
\]

and \(\alpha_0 = 0\). By a similar argument, if every entry of \(X \in S\) is contained in \([0, 1/m]\), then \(\alpha_0 = 1\).
In order to state a more general result, it is necessary to define some notation. Let $A_n$ be the diagonal idempotent matrix of rank $K$. Then

$$y_1 \cdots y_n = \begin{pmatrix} \sum_{j} W_{1jn} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \sum_{j} W_{mjn} & 0 & \cdots & 0 \end{pmatrix}$$

where $j \in \{1, \ldots, m^n - 1\}$ and $W$ represents the product of $n$ real numbers. We need to consider the distribution of $S_{n_1} = \sum_{j} W_{1jn}$ where $j \in \{1, 2, \ldots, m^n - 1\}$. Let $F_{1jn}$ be the distribution function of the random variable $W_{1jn}$.

**Theorem 2.** Suppose the $(W_{1jn})$ defined above satisfy the following conditions for each $i$:

1. $\sum_{j} \text{Var}(W_{1jn}) = 1$ for every $n$.
2. $E(W_{1jn}) = 0$ for every $j, n$.

If $\int y^2 \, dF_{1jn}(y) \to 0$ where the integral is taken over the set $|y^2| > \delta$ for each $\delta > 0$ as $n \to \infty$ then $\sigma_0 = 1$.

**Proof:** By the Lindeberg-Feller Theorem, $S_{n_1}$ converges in distribution to the standard normal for every $i$. Therefore for $n$ and $N$ sufficiently large,

$$P(|S_{n_1}| \leq N) = 1 - \varepsilon$$

for all $i$ where $\varepsilon \to 0$ as $N \to \infty$. Therefore

$$\mu(X = (x_{ij}); x_{i1} \cdots x_{in} \in K_k = [-k, k])$$

$$= \mu(X; \sum_{i} x_{ij} S_{n_1} \leq k \text{ for all } j)$$

$$\geq \mu(X; \sum_{i} x_{ij} |N \leq k \text{ for all } j)(1 - \varepsilon)^{m}$$

$$\geq (1 - \varepsilon)^{m} \mu(K_k)$$.

Note that $k$ depends only on the choice of $N$ and $K$ and not on the choice of $n$. Therefore as $K_k \uparrow S$ we may also let $N$ increase so it becomes clear that $\sigma_0 = \sup \lim \mu^{n}(Kd^{-1}) = 1$.

It is clear that conditions (1) and (2) may be relaxed so that

$$\sum_{j} \text{Var}(W_{1jn}) < M \text{ for some } M$$

and $E(W_{1jk}) < \infty$ for all $j, k$. QED
We present an example. Suppose the support of the measure $\mu$.

$$S_\mu = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}.$$  

Then for any $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in S_\mu$, $x_1 = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. 

Therefore we need only be concerned with the probability distribution of the corner element. Suppose $X_{ij}$ is a random variable such that $P(X_{ij} = 1/2) = P(X_{ij} = -1/2) = 1/2$ for all $i,j$.

Then $E(X_{ij}) = 0$ and $\text{Var}(X_{ij}) = 1/4$. Also for any $n$, $E(X_{ij}X_{2j} \cdots X_{nj}) = 0$ and $\text{Var}(X_{ij}X_{2j} \cdots X_{nj}) = 1/4$. Define

$$W_{1j1} = 2X_{1j}, \; j = 1$$
$$W_{1j2} = 2X_{1j}X_{2j}, \; j = 1,2,3,4$$
$$W_{1jn} = 2X_{1j}X_{2j} \cdots X_{nj}, \; j = 1,2,\ldots,4^{n-1}.$$  

Then $\text{Var}(W_{1jn}) = 1$ and $E(W_{1jn}) = 0$. Also $\int y^2 dF_{1jn}(y) = 0$ if $n$ is sufficiently large. By the above theorem, $a_0 = 1$.

4. The Case Where $a_0 = 1$.

If $a_0 = 0$ then for all compact sets $K$, lim sup $n \mu^n(Kx^{-1}) = 0$ so that it is clear that for any sequence $(a_n)$, $\mu^n = \delta_{a_n}$ converges vaguely to the zero measure.

Therefore we concentrate on the case where $a_0 = 1$. Let $S$ be a locally compact, second countable, Hausdorff semigroup satisfying condition (c).

**Lemma 2.** If $a_0 = 1$ and $S$ is abelian then there exists a sequence $(x_n)$ such that for any $0 \leq \alpha < 1$ there exists a compact set $K_\alpha$ such that $\mu^n(K_\alpha x_n^{-1}) > \alpha$ for all $n$.

**Proof:** For $\alpha = 1/2$ there exists a $K_2$ such that lim sup $\mu^n(K_2^{-1}) > 1/2$ for all $x \in S$. Therefore there exists a sequence $(x_n)$ such that $\mu^n(K_2^{-1}) > 1/2$ for all $n$. Similarly, for each $\alpha > 1/2$ there exists a $K_\alpha$ and a sequence $(x_n)$ such that $\mu^n(K_\alpha x_n^{-1}) > \alpha$. Since $\alpha > 1/2$, the sets $K_2^{-1}$ and $K_\alpha x_n^{-1}$ cannot be disjoint so there must exist $w \in (K_2^{-1}) \cap (K_\alpha x_n^{-1})$. This implies that
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\[ x_n \in K x^{-1} \subset K (K_a x^{-1})^{-1}. \] Therefore \( K x^{-1} \subset K (K_a x^{-1})^{-1}. \) Suppose
\[ y \in K x^{-1}. \] Then \( y \in K (K_a x^{-1})^{-1} \) so there exists \( z \in K (K_a x^{-1})^{-1} \) such that
\[ yz \in K_a. \] Also \( z \in K (K_a x^{-1})^{-1} \) implies there exists \( z' \in K_a \) such that
\[ zz' \in K_a \] and \( z' x_n \in K_a. \) Therefore \((yz)(z' x_n) \in K_a K_a. \) Since \( S \) is abelian,
\[ y x_n \in (K_a^{-1} x K_a) K_a \] and \( y \in ((K_a^{-1} x K_a)^{-1} x K_a x^{-1}). \) By redefining \( K_a \) to be
\[ (K_a^{-1} x K_a), \] \( \mu^n(K_a x^{-1}) > a \) for all \( n. \)

QED

If \( S \) is a group we can define \( \nu_n = \delta_n^n \ast \mu_n \ast \delta_x \) where the \( x_n \) are defined in lemma 2. Then we can apply Csizsar [1] to
\[ y^k = \nu_{k+1} \ast \nu_{k+2} \ast \ldots \ast \nu_n = \delta_{x_{n-1}} \ast \mu_{x_k} \ast \delta_{x_{n-1}}. \] Unfortunately \( \delta_n \) has no
meaning in a semigroup and the \( \nu_n \) must be defined in some other way.

Suppose \( S \) is embeddable in an abelian group \( G. \) Then by Lemma 2 there exists
a sequence \((x_n)\) such that for any \( a \) there exists \( K_a \) such that \( \mu^n(K_a x^{-1}) > a. \)

We may assume that \( \mu \) is a measure defined in \( G \) with support contained in \( S. \) Then
\[ \nu_n = \delta_n^n \ast \mu_n \ast \delta_{x_{n-1}} \] is well defined in \( G \) if we let \( x_0 \) be the identity element
of \( G. \) If we write \((Kx^{-1})_S\) and \((Kx^{-1})_G\) for the respective sets defined in \( S \) and \( G\)
then \((Kx^{-1})_S \subset (Kx^{-1})_G. \) However since the support of \( \mu \) is contained in \( S,
\[ \mu^n((Kx^{-1})_S) = \mu^n((Kx^{-1})_G). \] Therefore \( a_0 = 1 \) with respect to \( G. \) Let
\[ y^k = \nu_{k+1} \ast \ldots \ast \nu_n \] be \( y_0^n(K_a) = \mu^n(K_a x^{-1}) > a \) for any \( a. \) Also, by lemma 1,
\[ y^k_n(K_a^{-1} x K_a) \geq y_0^n(K_a) + y_0^n(K_a^{-1}) = 2a - 1. \] Therefore it is clear that any limit
point of \( y_k^n \) must be a probability measure and Csizsar [1] can be applied to this
sequence. It is also clear that any limit point of \( y_k^n \) must have support
contained in \( S \) and may therefore be considered a measure on \( S. \)

Next consider the case where \( S \) is an abelian inverse semigroup. \( S \) is a
semigroup of this type provided for any \( x \in S \) there exists a unique \( x' \in S \) such that
\( xx' = x \) and \( x' xx' = x. \) A natural ordering can be defined on the
idempotent elements of \( S: e \leq f \) provided \( ef = fe = e. \) If \( S \) contains a minimal
idempotent \( e \) then we can define \( \nu_n = \delta_{x_{n-1}} \ast \mu_n \ast \delta_{x_{n-1}} \) with \( x_0 = e. \) Then
\[ y_0^n(K_a \cup K_a e) = \nu_1 \ast \ldots \ast \nu_n(K_a \cup K_a e) \]
\[ = \mu^n((K_a \cup K_a e) x_{n-1}^{-1}) \]
\[ > \mu^n(K_a x_{n-1}^{-1}) > a \] for all \( n. \)

Therefore all limit points of \( y_k^n \) are probability measures.
If $S$ contains a finite number of idempotents, say $e_1, e_2, \ldots, e_n$ then the product $e_1 e_2 \cdots e_n$ is minimal in $S$. Therefore Csiszar [1] can be applied to any abelian inverse semigroup with a finite number of idempotents.

Suppose instead that $S$ is an inverse semigroup such that the set of idempotents can be ordered in the following manner: $f_0 > f_1 > f_2 > \cdots$. That is, suppose $S$ is an $\omega$-semigroup. Let $x_0 = f_0$ and consider the sequence $(x_n)$ defined in lemma 4. Given any $x_n$ either

a. the idempotent $x_j x_j^* = e_j$ for all $j > n$ or
b. there exists some $j > n$ such that $e_j < e_n$.

If there exists some $n$ for which (a) is true then $S$ has a minimal idempotent. If not, there exists a subsequence $x_0, x_1, x_2, \ldots$ such that $e_j > e_i$ if $j > n$.

Define $\nu_n = \delta_{x_n} = \mu_n^{-1} x_n^* \delta_{x_n^*} \mu_n^{-1}$.

**THEOREM 3.** If $y_k^n = \nu_{k+1} \cdot \cdots \cdot \nu_n$ is a sequence of probability measures on $S$ satisfying the hypotheses of Csiszar [1] then there exists a sequence $(\omega_n)$ in $S$ such that for each $K$, $y_k^n \Rightarrow \omega_n$ converges vaguely to a probability measure as $n \to \infty$.

**PROOF.** By Csiszar [1] there exists a sequence of integers $n_1 < n_2 < \cdots < n_j < \cdots$ such that

$$\lim_{j \to \infty} y_k^{n_j} = \lambda_k \quad \text{and} \quad \lim_{j \to \infty} \lambda_n = \lambda_\infty$$

where the limits are defined with respect to the vague topology and $\lambda_k$ is a probability measure for all $k \leq \infty$. Also $\lambda_\infty$ is idempotent and $\lambda_k \ast \lambda_\infty = \lambda_k$ for all $K$.

The support of any idempotent probability measure is completely simple. Let $H$ denote the support of $\lambda_\infty$. Since $S$ is abelian, $H$ is a group. Furthermore, $\lambda_\infty$ is a Haar measure on $H$ and $H$ is a compact group.

The remainder of the proof, dealing with the choice of a suitable sequence $(\omega_n)$, is quite similar to the argument in Csiszar [4] and will be omitted.

**QED**

We define $a_n = x_n \omega_n$ where $x_n$ is defined in lemma 2 and $\omega_n$ is defined above.

If $S$ is embeddable or an inverse semigroup with a minimal idempotent then

$$\lim_{n \to \infty} y_0^n = \delta_{\omega_n} = \lim_{n \to \infty} \mu_n \ast \delta_{x_n} = \lambda_0$$

which is a probability measure. In the other two cases, the same argument can be applied to an infinite subsequence.
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