A DIGRAPH EQUATION FOR HOMOMORPHIC IMAGES

ROBERT D. GIRSE and RICHARD A. GILLMAN
Department of Mathematics
Idaho State University
Pocatello, Idaho 83209

(Received November 20, 1985)

ABSTRACT. The definitions of a homomorphism and a contraction of a graph are
generalized to digraphs. Solutions are given to the graph equation $\phi(D) = \theta_\phi(D)$.

KEY WORDS AND PHRASES. Homomorphisms of graphs, contractions of graphs, digraphs.
1980 AMS SUBJECT CLASSIFICATION CODE. 05C20

By a graph $G$ we mean a finite graph with no multiple edges or loops. If graphs
$G$ and $H$ are isomorphic we write $G = H$. An elementary homomorphism of a graph $G$ is
an identification of two non-adjacent vertices of $G$ and a homomorphism is a sequence
of elementary homomorphisms. A homomorphism of $G$ onto $H$ preserves adjacency.
Likewise, an elementary contraction of $G$ is the identification of two adjacent
vertices of $G$ and a contraction is a sequence of elementary contractions\[1]. Thus
for every homomorphism $\phi$ of $G$ there is a related contraction $\theta_\phi$ of the complement
of $G$, $\overline{G}$. This contraction is constructed as follows: $\phi$ is a sequence of elementary
homomorphisms $e_1, e_2, \ldots, e_n$ so we let $\theta_\phi$ be sequence of elementary contractions
$\theta_1, \theta_2, \ldots, \theta_n$ where $\theta_i$ identifies the same vertices in $\overline{G}$ that $e_i$ identifies in $G$.

Recently [2] the graph equation $\phi(\overline{G}) = \theta_\phi(\overline{G})$ was studied. In this paper, we
generalize the definition of a homomorphism and its related contraction to digraphs
and find general solutions to this graph equation. In doing so, we find an easier
proof of the result given in [2].

A digraph $D$ consists of a finite vertex set $V(D)$ together with a set $E(D)$ of
ordered pairs of distinct elements of $V(D)$, called arcs. Again, if $D_1$ is isomorphic
to $D_2$ we write $D_1 = D_2$. By an elementary homomorphism of $D$ we mean an identification
of two mutually non-adjacent vertices of $D$ (neither uv nor vu are in $E(D)$).
Similarly, an elementary contraction is an identification of two mutually adjacent
vertices of $D$ (both uv and vu are in $E(D)$). A homomorphism (contraction) of $D$ is
again a sequence of elementary homomorphisms (contractions). The contraction
$\theta_\phi$ of $\overline{D}$ related to the homomorphism $\phi$ of $D$ is defined as for undirected graphs.
We will use the following notation as need arises: $I_b(u)$ is the set of vertices $v$ of $D$ such that $vu$ is an arc of $D$, $O_b(u)$ is the set of vertices $v$ of $D$ such that $uv$ is an arc of $D$, and $A(u)$ is the adjacency set of $u$ in the graph $G$.

**Theorem 1.** Let $\epsilon$ be an elementary homomorphism of $D$ identifying vertices $u_1$ and $u_2$. Then $\epsilon(D) = \theta_\epsilon(D)$ if and only if $I_b(u_1) = I_b(u_2)$ and $O_b(u_1) = O_b(u_2)$.

**Proof.** Let $u = \epsilon(u_1) = \theta_\epsilon(u_1)$. First suppose that $O_b(u_1) \neq O_b(u_2)$. Excluding $u$ as a possible endpoint of an arc, we have $vv'$ is an arc of $\epsilon(D)$ if and only if $vv'$ is an arc of $\theta_\epsilon(D)$. Hence there is a one to one correspondence of those arcs in $\epsilon(D)$ without $u$ as an endpoint and those of $\theta_\epsilon(D)$ without $u$ as an endpoint. The vertex $v$ of the arc $uv$ must be in $O_b(u_1) \cap O_b(u_2)$, $(O_b(u_1) \cup O_b(u_2))^c$, or $O_b(u_1) \setminus O_b(u_2)$, the symmetric difference. In the first case, $uv$ is not an arc of $\epsilon(D)$ or $\theta_\epsilon(D)$ and in the second case, $uv$ is an arc of both. The latter case implies that $uv$ is not an arc of $\epsilon(D)$ but is an arc of $\theta_\epsilon(D)$. Thus for every vertex in $O_b(u_1) \setminus O_b(u_2)$, $\theta_\epsilon(D)$ has one more arc than $\epsilon(D)$. The same holds for vertices in $I_b(u_1) \setminus I_b(u_2)$. Thus if $O_b(u_1) \neq O_b(u_2)$ or $I_b(u_1) \neq I_b(u_2)$, $|E(\theta_\epsilon(D))| > |E(\epsilon(D))|$ and hence $\epsilon(D) \neq \theta_\epsilon(D)$. Now let $I_b(u_1) = I_b(u_2)$ and $O_b(u_1) = O_b(u_2)$. We will use the identity map from $V(\epsilon(D))$ onto $V(\theta_\epsilon(D))$ and hence need only consider arcs to and from $u$. If $uv$ is in $E(\theta_\epsilon(D))$ then $u_1v$ and $u_2v$ are arcs in $\bar{D}$. Thus $u_1v$ and $u_2v$ are not arc of $D$ and subsequently $uv$ is in $E(\epsilon(D))$. By the same argument, if $uv$ is an arc of $\epsilon(D)$, $uv$ will be an arc of $\theta_\epsilon(D)$. This holds for arcs $vu$, so $\epsilon(D) = \theta_\epsilon(D)$.

**Corollary 1:** $\phi(D) = \theta_\phi(D)$ if and only if $\phi$ is a sequence of elementary homomorphisms, each of which satisfies the conditions of Theorem 1.

A digraph $D$ is pseudo-complete $n$-partite if there is a partition $V_1, V_2, \ldots, V_n$ such that $u, u'$ in $V_i$ for some $i$ implies $u$ and $u'$ are mutually non-adjacent, if $u$ is an element of $V_i$ and $v$ is an element of $V_j$, if $j$, then either $uv$ or $vu$ is an arc of $D$, and finally if $u$ and $u'$ are in $V_i, v$ and $v'$ are in $V_j$, if $j$, and $uv$ is an arc then $uv'$, $u'v$, and $u'v'$ are also.

**Theorem 2.** $\phi(D) = \theta_\phi(D)$ for all homomorphisms $\phi$ of $D$ if and only if $D$ is pseudo-complete $n$-partite.

**Proof.** If $D$ is pseudo-completely $n$-partite, every elementary homomorphism identifies two vertices $u_1$ and $u_2$ in the same partition set and thus $I_b(u_1) = I_b(u_2)$ and $O_b(u_1) = O_b(u_2)$. Hence $\epsilon(D) = \theta_\epsilon(D)$ for every elementary homomorphism and thus for every homomorphism of $D$. Conversely, partition $V(D)$ according to the relation: $u_1$ and $u_2$ are in $V_i$ if and only if $I_b(u_1) = I_b(u_2)$ and $O_b(u_1) = O_b(u_2)$. We need only show that if $u_1$ is in $V_i$ and $u_2$ is in $V_j$, if $j$, then either $u_1u_2$ or $u_2u_1$ is in $E(D)$. Suppose $u_1$ and $u_2$ are mutually non-adjacent and let $\epsilon$ be the elementary homomorphism identifying them. Since $\epsilon(D) = \theta_\epsilon(D)$, $O_b(u_1) = O_b(u_2)$ and $I_b(u_1) = I_b(u_2)$ by Theorem 1 and hence $u_1$ and $u_2$ are in the same partition set. Thus if $u_1$ is in $V_i$ and $u_2$ is in $V_j$, if $j$, there is an arc between them and $D$ must be pseudo-complete $n$-partite.
If, for every vertex \( u \) of \( D \), \( I_b(u) = O_b(u) \), \( D \) is a symmetric digraph and can be represented by a graph \( G \). This leads to the following corollaries to Theorems 1 and 2.

**COROLLARY 2.** An elementary homomorphism \( \epsilon \) identifying vertices \( u \) and \( v \) of a graph \( G \) satisfies \( \overline{\epsilon(G)} = \overline{\epsilon(G)} \) if and only if \( A(u_1) = A(u_2) \).

**COROLLARY 3.** A homomorphism \( \phi \) of \( G \) satisfies \( \overline{\phi(G)} = \overline{\phi(G)} \) if and only if \( \phi \) is a sequence of elementary homomorphisms, each satisfying Corollary 2.

**COROLLARY 4.** \( \overline{\phi(G)} = \overline{\phi(G)} \) for every homomorphism \( \phi \) of \( G \) if and only if \( G \) is complete \( n \)-partite.

A study of the equation \( \phi(D) = \overline{\phi(D)} \) would be interesting, yet is apparently difficult considering the work done in [2] for graphs. We conjecture that if \( D = \overline{D} \) and \( \phi(D) = \overline{\phi(D)} \), \( \phi \) nontrivial, then \( D \) is a symmetric digraph.

**REFERENCES**

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