ON SOME SPACES OF SUMMABLE SEQUENCES AND THEIR DUALS

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(Received May 18, 1984 and Revised March 11, 1985)

ABSTRACT. Suppose that S is the space of all summable sequences α with $\|\alpha\|_S = \sup_{n \geq 0} \left| \sum_{j=n}^{\infty} \alpha_j \right|$ and J the space of all sequences β of bounded variation with $\|\beta\|_J = |\beta_0| + \sum_{j=1}^{\infty} |\beta_j - \beta_{j-1}|$. Then for α in S and β in J $\left| \sum_{j=0}^{\infty} \alpha_j \beta_j \right| \leq \|\alpha\|_S \|\beta\|_J$; this inequality leads to the description of the dual space of S as J . It, related inequalities, and their consequences are the content of this paper. In particular, the inequality cited above leads directly to the Stolz form of Abel's theorem and provides a very simple argument. Also, some other sequence spaces are discussed.

KEY WORDS AND PHRASES. Dual spaces, sequence spaces, bounded operators.

1980 Mathematics Subject Classification: 46A45, 46A20

1. INTRODUCTION.

Suppose S is the space of all summable sequences $s = (s_n)_{n=0}^{\infty}$, with norm given by $\|s\|_S = \sup_{n \geq 0} \left| \sum_{j=n}^{\infty} s_j \right|$, the remainder in the sum of s after n -terms. An example of such an $\bar{s}$, summable but not absolutely summable, is given by $s_n = \frac{(-1)^n}{n+1}$, $n = 0, 1, 2, 3, \dots$, so that $\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1}$ is an alternating convergent series. Here, as is readily verified, $\|s\|_S = \sup_{n \geq 0} \left| \sum_{j=n}^{\infty} \frac{(-1)^j}{j+1} \right| = \ln 2$, the natural log of 2. This paper is based on the observation that S , with this norm, is naturally isomorphic to c_0 , the space of all sequences having limit zero endowed with the supremum norm. Moreover, if we consider S on its own, we get several interesting results. For example, since S is isomorphic to c_0 and c_0^* , the dual of c_0 , is isomorphic to ℓ_1 , the space of all absolutely summable sequences, then it follows that S^* , the dual of S , is isomorphic to ℓ_1 .

However, if we compute the dual of S without using the isomorphism with c_0 , we find that S^* is isomorphic to J , the space of all sequences β of bounded variation with

$$\|\beta\|_J = |\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|; \text{ an example of such a } \beta \text{ being given}$$

$$\text{by } \beta_n = \frac{1}{n+1} \quad n = 0, 1, 2, 3, \dots \text{ so that } \|\beta\|_J = 1 + \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right] = 2.$$

Of course, J is isomorphic to ℓ_1 . Another example is that we get the following

new inequality: If $\beta \in J$ and $s \in S$ $\left| \sum_{j=0}^{\infty} s_j \beta_j \right| < \|s\|_S \cdot \|\beta\|_J$, which can be used to give a proof of Abel's Continuity Theorem.

Similarly, we consider the space H of all sequences $\lambda = (\lambda_n)_{n=0}^{\infty}$ such that

$$\|\lambda\|_H = \sup_{n>0} \left| \sum_{j=0}^n \lambda_j \right| \text{ is finite; for example } \lambda_n = (-1)^n, n = 0, 1, 2, 3, \dots, \text{ so}$$

that $\|\lambda\|_H = 1$. We get the following inequality: If $\lambda \in H$, $\beta \in J$, and $\rho = \lim \beta$

$$\text{then } \left| \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \right| \leq \|\lambda\|_H \cdot \|\beta\|_J. \text{ Consequently we have that } \phi \in J^* \text{ if}$$

$$\text{and only if } \phi(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \text{ for } \beta \in J.$$

Also, we give characterizations of the bounded linear operators on S easily obtained from a description of operators on c_0 . These results, however, may be obtained using the uniform boundedness principle.

To make the presentation reasonably self-contained, we shall include a resume of pertinent results and definitions.

2. PRELIMINARIES

DEFINITION 2.1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two sequence spaces, which are Banach spaces with the respective norms. For a fixed $y = (y_n) \in Y$ define the mapping $\phi_y: X \rightarrow \mathbb{R}$ by $\phi_y(x) = \sum_{n=0}^{\infty} x_n y_n$ for those $x = (x_n)$ in X for which the series converges.

Assume that $|\phi_y(x)| \leq M \|x\|_X$ with M some absolute constant, that is, ϕ_y is a bounded linear functional on X for any fixed $y \in Y$. Let X^* be the space of all bounded linear functionals ϕ on X , endowed with norm $\|\phi\|_{X^*} = \sup_{\|x\|_X=1} |\phi(x)|$ and ψ the mapping from Y to X^* given by

$\psi(y) = \phi_y$. If ψ is onto and $\|y\|_Y = \|\phi_y\|_{X^*}$, then we say that Y is the dual of X , in this sense we write $Y = X^*$.

DEFINITION 2.2. We define the sequence spaces S , ℓ_1 , ℓ_{∞} , c_0 , and H respectively by

$$S = \{s = (s_n)_{n=0}^{\infty}, \text{ summable sequences such that } \|s\|_S = \sup_{p \geq 0} \left| \sum_{j=p}^{\infty} s_j \right|\},$$

$$\ell_1 = \{\delta = (\delta_n)_{n=0}^{\infty}, \|\delta\|_{\ell_1} = \sum_{j=0}^{\infty} |\delta_j| < \infty\},$$

$$\ell_{\infty} = \{\mu = (\mu_n)_{n=0}^{\infty}, \|\mu\|_{\ell_{\infty}} = \sup_{j \geq 0} |\mu_j| < \infty\},$$

$$c_0 = \{\gamma = (\gamma_n)_{n=0}^{\infty}, \lim_{n \rightarrow \infty} \gamma_n = 0, \|\gamma\|_{c_0} = \sup_{n \geq 0} |\gamma_n|\}$$

$$H = \{ \lambda = (\lambda_n)_{n=0}^{\infty}, \| \lambda \|_H = \sup_{n \geq 0} \left| \sum_{j=0}^n \lambda_j \right| < \infty \}.$$

We believe that this is the first time that S is taken as a space with this norm.

THEOREM 2.3. The space S is a Banach space, isometrically isomorphic to c_0 . The isometry $T: S \rightarrow c_0$ being given by $T(s) = (r_n(s))_{n=0}^{\infty}$ and $T^{-1}: c_0 \rightarrow S$ by $T^{-1}(\gamma) = (\gamma_n - \gamma_{n+1})_{n=0}^{\infty}$, where $r_n(s) = \sum_{j=n}^{\infty} \alpha_j$.

DEFINITION 2.4. Two Banach spaces $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are isomorphic equivalents if there exist a one-to-one and onto linear mapping $T: X \rightarrow Y$ such that $N \| x \|_X \leq \| Tx \|_Y \leq M \| x \|_X$ with N and M absolute constants. If there exists such a T with $\| Tx \|_Y = \| x \|_X$, then we say that X and Y are isometrically isomorphic.

Notice in Theorem 2.3 T and T^{-1} can be represented as infinite matrices,

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 0 & 1 & -1 & 0 \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

THEOREM 2.5. M is a bounded linear operator from c_0 into c_0 if and only if M is represented by an infinite matrix with columns in c_0 and rows in ℓ_1 , uniformly bounded in ℓ_1 . That is if $M = (m_{nk})$ then

- i) $\lim_{n \rightarrow \infty} m_{nk} = 0$ for any fixed k ,
- ii) $\sup_{n \geq 0} \sum_{k=0}^{\infty} |m_{nk}| < \infty$.

THEOREM 2.6. ϕ is in c_0^* if and only if there is a $\delta = (\delta_n) \in \ell_1$ such that for any $\gamma = (\gamma_n) \in c_0$ $\phi(\gamma) = \sum_{n=0}^{\infty} \delta_n \gamma_n$, in which case there is only one such $\delta = (\delta_n) \in \ell_1$ and $\| \phi \|_{c_0^*} = \| \delta \|_{\ell_1}$.

THEOREM 2.7. The space J is a Banach space, isometrically isomorphic to ℓ_1 . The isometry $L: J \rightarrow \ell_1$ is given by $L(\beta) = (\beta_n - \beta_{n-1})_{n=0}^\infty$,

$\beta_{-1} = 0$, and $L^{-1}: \ell_1 \rightarrow J$ by $L^{-1}(\delta) = (\sum_{j=0}^n \delta_j)_{n=0}^\infty$ with $\delta = (\delta_n) \in \ell_1$ and $\beta = (\beta_n) \in J$.

THEOREM 2.8. The space H is a Banach space isometrically isomorphic to ℓ_∞ .

The isometry $\psi: H \rightarrow \ell_\infty$ is given by $\psi(\lambda) = (\sum_{j=0}^n \lambda_j)_{n=0}^\infty$ and $\psi^{-1}: \ell_\infty \rightarrow H$

by $\psi^{-1}(\mu) = (\mu_n - \mu_{n-1})_{n=0}^\infty$ with $\mu_{-1} = 0$, $\mu = (\mu_n) \in \ell_\infty$, $\lambda = (\lambda_n) \in H$.

3. THE DUAL OF S

THEOREM 3.1 (Holder's type inequality). If $s = (s_n) \in S$ and $\beta = (\beta_n) \in J$

then $\left| \sum_{n=0}^\infty s_n \beta_n \right| \leq \| \beta \|_J \cdot \| s \|_S$.

Proof: For $N \geq 1$, we notice that

$$\left| \sum_{n=0}^{N-1} s_n \beta_n - r_0(s) \beta_0 - \sum_{n=1}^N r_n(s) [\beta_n - \beta_{n-1}] \right| = |r_N(s) \beta_N| = |r_N(s)| \cdot |\beta_N|,$$

where $r_n(s) = \sum_{j=n}^\infty s_j$. Now since s is summable and β is bounded, it

follows that $|r_N(s)| \cdot |\beta_N| \rightarrow 0$ as $N \rightarrow \infty$. Therefore we have proved that

$$\sum_{n=0}^\infty s_n \beta_n = r_0(s) \beta_0 + \sum_{n=1}^\infty r_n(s) [\beta_n - \beta_{n-1}]; \text{ consequently}$$

$$\left| \sum_{n=0}^\infty s_n \beta_n \right| \leq \sup_{n \geq 0} |r_n(s)| (|\beta_0| + \sum_{n=1}^\infty |\beta_n - \beta_{n-1}|),$$

so that the theorem is proved.

Consider the mapping $\phi_\beta: S \rightarrow \mathbb{R}$ defined by $\phi_\beta(s) = \sum_{n=0}^\infty s_n \beta_n$ where $\beta = (\beta_n)$ is a fixed element in J . In view of Theorem 3.1, we have ϕ_β is a bounded linear functional on S . At this point, a natural question is: are these all the bounded linear functionals on S ? The next result tells us that the answer is yes.

THEOREM 3.2. (Duality Theorem). If $\phi \in S^*$ then there is a unique $\beta = (\beta_n) \in J$ such that $\phi = \phi_\beta$, that is, $\phi(s) = \sum_{n=0}^\infty s_n \beta_n$ for any $s = (s_n) \in S$. Moreover

$\| \phi \|_{S^*} = \| \beta \|_J$. Conversely if $\phi(s) = \sum_{n=0}^\infty s_n \beta_n$ then $\phi \in S^*$ so that the mapping $\psi: J \rightarrow S^*$ defined by $\psi(\beta) = \phi_\beta$ is an isometric isomorphism from J to S^* .

Proof: If $\phi(s) = \sum_{n=0}^\infty s_n \beta_n$, then we already have seen by Theorem 3.1 that ϕ is a bounded linear functional on S , that is $\phi \in S^*$. So it remains to prove the first part. Let $\phi \in S^*$; then using Theorem 2.3, we have

$$\phi(s) = \phi(T^{-1} \circ T(s)) = (\phi \circ T^{-1})(T(s)) = (\phi \circ T^{-1})\left(\left(\sum_{j=n}^\infty s_j\right)_{n=0}^\infty\right).$$

Now notice that $\phi \circ T^{-1} \in c_0^*$, so that there is a $y = (y_n) \in \ell_1$ (see Theorem 2.6) such that

$$\phi(s) = (\phi \circ T^{-1})\left(\left(\sum_{j=n}^{\infty} s_j\right)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} y_n \left(\sum_{j=n}^{\infty} s_j\right) \quad \text{and} \quad \|\phi \circ T^{-1}\|_{c_0^*} = \|y\|_{\ell_1}. \quad (3.3)$$

Observe that (3.3) can be written $\phi(s) = \sum_{n=0}^{\infty} s_n \sum_{j=0}^n y_j$; therefore we may let $\beta = (\beta_n)$

$$\text{where } \beta_n = \sum_{j=0}^n y_j.$$

Since $\beta = L^{-1}(y)$, β is in J . Consequently given $\phi \in S^*$ there is a $\beta \in J$ such that $\phi = \phi_\beta$. On the other hand Theorem 2.6 tells us that

$$\|\phi \circ T^{-1}\|_{c_0^*} = \sum_{n=0}^{\infty} |y_n|; \text{ therefore } \|\beta\|_J = \sum_{n=0}^{\infty} |y_n| = \|\phi \circ T^{-1}\|_{c_0^*} \leq \|T^{-1}\| \cdot \|\phi\|_{S^*}.$$

Now as $\|T^{-1}\| \leq 1$, it follows that $\|\beta\|_J \leq \|\phi\|_{S^*}$. Also notice that Theorem 3.1 implies $\|\phi\|_{S^*} \leq \|\beta\|_J$ (since $\phi = \phi_\beta$). Therefore putting together these last two inequalities we have $\|\phi\|_{S^*} = \|\beta\|_J$; consequently the mapping $\psi: J \rightarrow S^*$ defined by $\psi(\beta) = \phi_\beta$ is an isometry, so that the theorem is proved.

We have used Theorem 3.1 to characterize all bounded linear functionals on S . Now we are going to use this theorem again to give a proof of Abel's continuity theorem which is as follows.

THEOREM 3.4 (Abel's continuity theorem). If $\alpha = (\alpha_n)$ is summable and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \text{ then } \lim_{z \rightarrow 1} f(z) = f(1), \text{ where } z \text{ is restricted to approach the point } 1 \text{ in such a way that } |z| < 1 \text{ and } \frac{1-z}{1-|z|} \text{ remains bounded.}$$

Proof: First of all, let C be a positive absolute constant such that

$$\frac{1-|z|}{1-|z|} \leq C \text{ and } |z| < 1. \text{ Notice that } \left| \sum_{p=N}^{\infty} \alpha_p z^p \right| \leq \|(z^p)_{p=N}^{\infty}\|_J \cdot \|(\alpha_p)_{p=N}^{\infty}\|_S$$

for $N \geq 1$ by the above inequality applied to the sequences (α_p) and (z^p) ,

since $(z^p)_{p=N}^{\infty}$ is in J ; in fact, since $|z| < 1$

$$\begin{aligned} \|(z^p)_{p=N}^{\infty}\|_J &= |z|^N + |z^N - z^{N+1}| + |z^{N+1} - z^{N+2}| + \dots = \\ &= |z|^N + |1-z|(|z|^N + |z|^{N+1} + \dots) = |z|^N \left(1 + \frac{1-|z|}{1-|z|}\right). \end{aligned}$$

Now using the hypothesis we have $\|(z^p)_{p=N}^{\infty}\|_J \leq 1 + C$. Consequently,

$$\left| \sum_{p=N}^{\infty} \alpha_p z^p \right| \leq (1+C) \|(\alpha_p)_{p=N}^{\infty}\|_S \rightarrow 0 \text{ as } N \rightarrow \infty \text{ since } (\alpha_n) \in S. \text{ Then}$$

$$\left| \sum_{p=0}^{\infty} \alpha_p z^p - \sum_{p=0}^{\infty} \alpha_p \right| \leq \left| \sum_{p=0}^{N-1} \alpha_p z^p - \sum_{p=0}^{N-1} \alpha_p \right| + \left| \sum_{p=N}^{\infty} \alpha_p z^p - \sum_{p=N}^{\infty} \alpha_p \right| = A+B. \text{ For } \epsilon > 0,$$

fix N so that $B \leq (2+C) \|(\alpha_p)_{p=N}^{\infty}\|_S \leq \epsilon/2$. For $|z-1|$ sufficiently small,

$A < \epsilon/2$. Hence $\lim_{z \rightarrow 1} \sum_{p=0}^{\infty} \alpha_p z^p = \sum_{p=0}^{\infty} \alpha_p$. Therefore the theorem is proved.

4. THE DUAL OF J

Of course, for fixed $s = (s_n)$ in S the formula $\phi_s(\beta) = \sum_{j=0}^{\infty} s_j \beta_j$ defines a bounded linear functional on J . However, these are not all the members of J^* . As a key to our description of J^* , we note that if

$\beta = (\beta_n)$ is in J with limit ρ then $\phi_s(\beta) = (\sum_{j=0}^{\infty} s_j) \rho + \sum_{j=0}^{\infty} s_j (\beta_j - \rho) = \lambda_0 \rho + \sum_{j=1}^{\infty} \lambda_j (\rho - \beta_{j-1})$, where $\lambda_0 = \sum_{j=0}^{\infty} s_j$ and for $j \geq 1$ $\lambda_j = -s_{j-1}$. Here $\sum_{j=0}^{\infty} \lambda_j = 0$; so that λ is in S .

THEOREM 4.1 (Holder's type inequality). If $\lambda = (\lambda_n) \in H$, $\beta = (\beta_n) \in J$, and

$$\rho = \lim \beta \quad \text{then} \quad \left| \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \right| \leq \|\lambda\|_H \cdot \|\beta\|_J.$$

Proof: This inequality follows easily from the observation that

$$\lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) = \lambda_0 \beta_0 + \sum_{n=1}^{\infty} \left(\sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1}); \text{ so that}$$

$$\left| \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \right| \leq \sup_{n \geq 0} \left| \sum_{j=0}^n \lambda_j \right| \left(|\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| \right).$$

As we did for S , we consider the mapping $\phi_\lambda: J \rightarrow \mathbb{R}$ defined

$$\text{by } \phi_\lambda(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \text{ where } \lambda = (\lambda_n) \text{ is a fixed element in } H.$$

Theorem 4.1 assures us that ϕ_λ is a bounded linear functional on J .

The claim of the next theorem is that these are all the bounded linear functionals on J . Here, our representation is in a slightly different sense than that of the Definition 2.1, the latter appearing more natural to us.

THEOREM (4.2) (Duality Theorem). If $\phi \in J^*$ then there is a unique

$\lambda = (\lambda_n) \in H$ such that $\phi = \phi_\lambda$, that is, $\phi(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1})$ for any $\beta = (\beta_n) \in J$. Moreover $\|\phi\|_{J^*} = \|\lambda\|_H$. Conversely if $\phi = \phi_\lambda$ then $\phi \in J^*$. So that the mapping $\psi: H \rightarrow J^*$ defined by $\psi(\lambda) = \phi_\lambda$ is an isometric isomorphism. Here, $\rho = \lim \beta$.

Proof: By the observation preceding the theorem we see that it remains to prove the first part. Let $\phi \in J^*$; then in fact using Theorem 2.7

$$\phi(\beta) = \phi(L^{-1} \circ L(\beta)) = (\phi \circ L^{-1})(L(\beta)) = (\phi \circ L^{-1})((\beta_n - \beta_{n-1})_{n=0}^{\infty}), \beta_{-1} = 0.$$

Now notice that $\phi \circ L^{-1} \in \ell_1^*$, so that there is a $\mu = (\mu_n) \in \ell_\infty$ such that

$$(4.3) \quad \phi(\beta) = (\phi \circ L^{-1})((\beta_n - \beta_{n-1})_{n=0}^{\infty}) = \sum_{n=0}^{\infty} \mu_n (\beta_n - \beta_{n-1}).$$

By Theorem 2.8 $\mu_n = \sum_{j=0}^n \lambda_j$ for $n \geq 0$ and some $\lambda = (\lambda_j) \in H$;

$$\text{consequently (4.3) becomes } \phi(\beta) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1}).$$

Observe now that

$$\sum_{n=0}^N \left(\sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1}) = \lambda_0 \beta_0 + \sum_{n=1}^N \left(\sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1}) =$$

$$\begin{aligned}
&= \lambda_0 \beta_N + \sum_{n=1}^N \lambda_n [\beta_N - \beta_{n-1}] = \lambda_0 \beta_N + \sum_{n=1}^N \lambda_n [\beta_N - \rho + \rho - \beta_{n-1}] = \\
&= \lambda_0 \rho + [\beta_N - \rho] \sum_{n=0}^N \lambda_n + \sum_{n=1}^N \lambda_n [\rho - \beta_{n-1}]. \text{ Taking the limit as } N \rightarrow \infty
\end{aligned}$$

we get

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \lambda_j \right) (\beta_n - \beta_{n-1}) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n [\rho - \beta_{n-1}]$$

since $[\beta_N - \rho] \sum_{n=0}^N \lambda_n \rightarrow 0$ as $N \rightarrow \infty$ ($\rho = \lim \beta$). Therefore we have

$$\phi(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n [\rho - \beta_{n-1}].$$

Again by Theorem 2.8 $\|\phi \circ L^{-1}\|_{\ell_1^*} = \|\mu\|_{\ell_\infty} = \sup_{n \geq 0} |\mu_n| = \sup_{j=0}^n \left| \sum_{j=0}^n \lambda_j \right| = \|\lambda\|_H$ or $\|\lambda\|_H = \|\phi \circ L^{-1}\|_{\ell_1^*} \leq \|\phi\|_{J^*} \cdot \|L^{-1}\| \leq \|\phi\|_{J^*}$ since $\|L^{-1}\| \leq 1$. On the other hand Theorem 4.1 gives us that $|\phi(\beta)| \leq \|\lambda\|_H \|\beta\|_J$ (since $\phi = \phi_\lambda$), which implies $\|\phi\|_{J^*} \leq \|\lambda\|_H$. Putting together these two inequalities we have $\|\phi\|_{J^*} = \|\lambda\|_H$, so that the theorem is proved.

5. CHARACTERIZATION OF OPERATORS ON S

In this section we characterize all bounded linear operators on S , even though an alternate proof can be given using the uniform boundedness principle. We rather use the characterization of operators on c_0 . In fact we have the following result.

THEOREM 5.1. The linear mapping $A: S \rightarrow c_0$ is bounded if and only if there

is an infinite matrix (a_{nk}) such that for any $s = (s_n) \in S$

$$As = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} & \dots \\ a_{10} & a_{11} & a_{12} & \dots & a_{1n} & \dots \\ \vdots & & & & & \\ \vdots & & & & & \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} & \dots \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \\ \vdots \end{bmatrix}$$

satisfying

i) $\lim_{n \rightarrow \infty} a_{nk} = 0$, for any fixed $k \geq 0$ i.e. the columns of the matrix are in c_0 ,

ii) $\sup_{n \geq 0} \sum_{k=0}^{\infty} |a_{n(k-1)} - a_{nk}| < \infty$ i.e. the rows of the matrix are uniformly bounded in J .

Here $a_{n(-1)} = 0$ for $n = 0, 1, 2, 3, \dots$.

Proof: Note that the operator $P = \begin{matrix} AT^{-1} \\ T^{-1}A \end{matrix}$ where T^{-1} is as in Theorem 2.3 maps c_0 into c_0 , that is $P: c_0 \rightarrow S \rightarrow c_0$, so that applying Theorem 2.5 we get the desired result.

THEOREM 5.2. The linear mapping $B: S \rightarrow S$ is bounded if and only if B is represented by an infinite matrix such that for some $A: S \rightarrow c_0$ we have

$B = A - \tilde{A}$, where $\tilde{A}$ is defined by $\tilde{A} = (\tilde{a}_{nk})$ with $\tilde{a}_{nk} = a_{(n+1)k}$.

($\tilde{A}$ is a shifting of A up by one row) Note that $a_{nk} = \sum_{j=n}^{\infty} b_{jk}$.

Proof: We define $A: S \xrightarrow{B} S \xrightarrow{T} c_0$, where T is as in Theorem 2.3. That is, $A = TB$ so that $B = T^{-1}A$.

Now using the matrix representation for T^{-1} we get

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad A = [I - \tilde{I}]A = IA - \tilde{I}A = A - \tilde{A} \quad \text{where}$$

I is the identity matrix and $\tilde{I}$ is formed by the shifting of I up by one row.

Notice Theorem 5.2 is equivalent to saying that $B: S \rightarrow S$ is a bounded operator if B is an infinite matrix such that each column of B is summable, and if we consider the matrix $r(B)$ with $r(B) = (y_{nk})$, where $y_{nk} = \sum_{j=n}^{\infty} b_{jk}$, then the rows of $r(B)$ are uniformly bounded in J .

THEOREM 5.3. The linear mapping $C: c_0 \rightarrow S$ is bounded if and only if C is represented by an infinite matrix such that for some $M: c_0 \rightarrow c_0$ we have $C = M - \tilde{M}$, $\tilde{M}$ is a shifting of M up by one row.

Proof: Notice $c_0 \xrightarrow{C} S \xrightarrow{T} c_0$, T as in Theorem 2.3. So define $M = TC$; therefore $C = T^{-1}M = M - \tilde{M}$.

Note here that the M 's are special A 's as are the B 's. The C 's are special B 's. The reader might wish to compare these last three theorems with Exercises 45 and 46, pg. 77, of [1].

We would like to point out that the spaces H and J may also be found in [1] (pg. 240). Another space there, denoted by cs , is the space of all

summable sequences $s = (s_j)$ normed with norm on H , $\max_{n \geq 0} \left| \sum_{j=0}^n s_j \right|$. It may be shown, as pointed out there, that cs^* looks like J . We hasten here to note that J is not the dual of cs in the sense of Definition 2.1. In particular,

the inequality $\left| \sum_{j=0}^{\infty} s_j \beta_j \right| < \|s\|_{cs} \cdot \|\beta\|_J$ does not hold. Consider the example:

Let $s = (s_n)$ be given by

$s_0 = 1, s_1 = s_2 = s_3 = s_4 = 0, s_5 = -2, s_6 = s_7 = \dots = 0$; also, $\beta = (\beta_n)$:

$\beta_0 = 0, \beta_1 = -1, \beta_2 = -2, \beta_3 = -3, \beta_4 = -4, \beta_5 = \beta_6 = \dots = -4$.

Notice that $\sum_{j=0}^{\infty} s_j \beta_j = 8, \|\beta\|_J = 4$ and $\|s\|_{cs} = \max_{n \geq 0} \left| \sum_{j=0}^n s_j \right| = 1$. Therefore

$\left| \sum_{j=0}^{\infty} s_j \beta_j \right| \leq \left[\max_{n \geq 0} \left| \sum_{j=0}^n s_j \right| \right] \cdot \|\beta\|_J$ does not hold.

To illustrate further the difference between cs and S , we note that cs and S are equivalent Banach spaces: if s is in S then $\|s\|_{cs} \leq 2\|s\|_S$

and $\|s\|_S \leq 2\|s\|_{cs}$. So that ϕ is in S^* if and only if ϕ is in cs^* .

For $j \geq 0$ let e_j be the sequence τ such that $\tau_j = 1$ and if $k \neq j$ $\tau_k = 0$.

Now for ϕ in S^* , $\phi(s) = \sum_{j=0}^{\infty} s_j \phi(e_j)$; the sequence $\{\phi(e_j)\}_{j=0}^{\infty}$ is in J and $\|\{\phi(e_j)\}\|_J = \|\phi\|_{S^*}$. But if β is the sequence of the example and ϕ is defined on S by $\phi(s) = \sum_{j=0}^{\infty} \beta_j s_j$ (so that $\beta_j = \phi(e_j)$) then $\|\beta\|_J = 4$. So the natural association of ϕ in cs^* with the sequence $\{\phi(e_j)\}$ in J is not an isometry. There is an isometry from cs^* onto J , obtained by noting that cs is isometrically isomorphic to the space c of all convergent sequences normed with norm in ℓ_{∞} ; the dual c^* being isometrically isomorphic to ℓ_1 . This representation of c^* as ℓ_1 , and also the representation of cs^* as J , is complicated by the fact that the sequence $\{e_j\}$ does not have dense span in c . However, the latter representation (of cs^* as J) is made somewhat simpler by applying Theorems 2.7 and 3.5. For s in S let $S_N(s) = \sum_{j=0}^N s_j$, the N th partial sum of s , and $\sigma(s) = \sum_{j=0}^{\infty} s_j$. Let ϕ be in cs^* with $\phi(e_j) = \beta_j$. Now, with L as in Theorem 2.7, we have $\phi(s) = \sum_{j=0}^{\infty} s_j \phi(e_j) = \sum_{j=0}^{\infty} \beta_j s_j = \beta_0 r_0(s) + \sum_{n=1}^{\infty} r_n(s) [\beta_n - \beta_{n-1}] = \beta_0 \sigma(s) + \sum_{n=1}^{\infty} [\sigma(s) - S_{n-1}(s)] [\beta_n - \beta_{n-1}] = \sigma(s) \nu_0 + \sum_{n=1}^{\infty} S_{n-1}(s) \nu_n$ ($\nu_0 = \lim \beta$, $\nu_n = \beta_{n-1} - \beta_n$); so $\nu = L(\nu)$, with $\mu_0 = \lim \beta$, $\mu_n = \lim \beta + \beta_0 - \beta_n$. These computations suggest that if one uses (as in [1]) the space cs rather than S the norm to be used in J should be given by $\|\beta\| = \left| \lim \beta \right| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|$. However, the norm used there is the same as ours (see 1, pg. 239). With this norm, $\|\beta\|$, on J , the Holder's type inequality we get is $\left| \sum_{j=0}^{\infty} \alpha_j \beta_j \right| \leq \|\alpha\|_{cs} \|\beta\|$, α in S and β in J .

The reader might wish to compare the remarks of this last paragraph with some of D. J. H. Garling in pages 999-1000 of [2] and page 964 of [3].

These ideas were suggested by a norm on certain power series spaces introduced by the latter author in an investigation [4] of Fourier sine series and Lidstone series.

REFERENCES

- [1] DUNFORD, N. and SCHWARTZ, J. T., Linear Operators, Part I, Interscience, New York, 1958.
- [2] GARLING, D. J. H., On Topological Sequences Spaces, Proc. Camb. Phil. Soc. 63 (1967), 997-1019.
- [3] GARLING, D. J. H., The β and γ -duality of Sequences Spaces, Proc. Camb. Phil. Soc. 63 (1967), 963-981.
- [4] GOLIGHTLY, George O., Sine Series on $[0, \ell]$ for Certain Entire Functions and Lidstone Series - To appear in J. Math. Anal and Applica.
- [5] ZYGMUND, A., Trigonometric Series, Cambridge University Press, London, 1959.