CHARACTER INDUCTION IN P-GROUPS

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ABSTRACT. Let G be a finite p -group and let χ be an irreducible character of G . Then χ is monomial; that is, $\chi = \lambda^G$, where λ is a linear character of some subgroup of G . We are interested in locating subgroups of G which induce the character χ .

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1. INTRODUCTION

For G a finite p -group and $\chi \in \text{Irr}(G)$ (the irreducible characters of G), χ non-linear ($\chi(1) \neq 1$) it is known that there is some subgroup H of G and some linear character $\lambda \in \text{Irr}(H)$ such that $\chi = \lambda^G$. We say χ is induced by λ . In this paper we find a way of locating proper subgroups of G which have a character that induces χ .

The notation in this paper follows that used in Isaacs [1]. The symbol $\phi(G)$ will denote the *Frattini subgroup* of G , the intersection of all maximal subgroups of G . For χ a character of G , $V(\chi) = \langle g \in G : \chi(g) \neq 0 \rangle$ is called the *support group* of χ and is the smallest subgroup of G outside of which χ vanishes. If N is a normal subgroup of G and $\psi \in \text{Irr}(N)$, then $I_G(\psi) = \{g \in G : \psi^g = \psi\}$ is the *inertia group* of ψ in G . If ψ is an irreducible constituent of χ_N then we know there is some $\theta \in \text{Irr}(I_G(\psi))$ such that $\theta^G = \chi$. The main result of this paper is the following:

THEOREM 1.1: Let G be a finite p -group and let χ be a non-linear irreducible character of G . Let N be a normal subgroup of G such that $V(\chi) \leq N \leq V(\chi)\phi(G)$ and let ψ be an irreducible constituent of χ_N . If ψ is non-linear then $I_G(\psi) < G$.

This theorem enables us , by induction on the order of G , to form chains of subgroups with associated characters. Each of these characters induces χ .

2. PRELIMINARIES

Besides Clifford's Theorem, Frobenius Reciprocity and the other fundamentals of character theory we will need the following results. The first is a corollary to a theorem of Isaacs[2]:

PROPOSITION 2.1 : Let N be a normal subgroup of G , $|G : N| = p$, p a prime. Suppose $\chi \in \text{Irr}(G)$. Then either

$$\text{a) } \chi_N \in \text{Irr}(N)$$

$$\text{or b) } \chi_N = \sum_{i=1}^p \theta_i \text{ where } \theta_i \text{ are distinct irreducible characters of } N$$

Let $\theta \in \text{Irr}(N)$. Then either

$$\text{a) } \theta^G = \sum_{i=1}^p \chi_i \text{ where } \chi_i \text{ are distinct irreducible characters of } G$$

$$\text{or b) } \theta^G \in \text{Irr}(G)$$

Futhermore, if ϕ is an irreducible constituent of χ_N and χ satisfies a (respectively b) of the first part then ϕ satisfies a (respectively b) of the second part. If ψ is an irreducible constituent of θ^G and θ satisfies a (respectively b) of the second part then ψ satisfies a (respectively b) of the first part.

LEMMA 2.2: Let χ be a non-linear irreducible character of G . Let N be a normal subgroup of G with $|G : N| = p$, p a prime , and $N \not\geq V(\chi)$. If ψ is an irreducible

constituent of χ_N , then $\psi^G = \chi$ and $\chi_N = \sum_{i=1}^p \psi_i$, where $\psi_i \in \text{Irr}(N)$ are distinct.

PROOF: The fact that ψ is a constituent of χ_N implies that χ is a constituent of ψ^G by Frobenius Reciprocity. Suppose $\theta \in \text{Irr}(G)$ such that θ is a constituent of ψ^G . Then ψ is also a constituent of θ_N , thus $[\chi_N, \theta_N] \neq 0$. Since $N \not\geq V(\chi)$, χ vanishes outside of N . Thus, by definition of inner product, we have

$$|G| [\chi, \theta] = \sum_{g \in G} \chi(g) \theta(g^{-1}) = \sum_{g \in N} \chi(g) \theta(g^{-1}) = |N| [\chi_N, \theta_N]. \quad (2.1)$$

Hence $[\chi, \theta] \neq 0$ yielding $\chi = \theta$. By lemma (2.1)(b) we have $\psi^G = \chi$ and $\chi_N = \sum_{i=1}^p \psi_i //$

PROPOSITION 2.3: Let G be a p -group with a non-linear irreducible character χ . Let θ be an irreducible constituent of $\chi_{V(\chi)}$. If $\theta(1) \neq 1$, then $I_G(\theta) < G$.

PROOF: Assume $\theta(1) \neq 1$ satisfies the above hypotheses. Now $\theta = \lambda^{V(\chi)}$ where λ is a linear character of some subgroup H of $V(\chi)$. Let M be a maximal subgroup of $V(\chi)$ containing H . Then $\theta = (\lambda^M)^{V(\chi)}$ by transitivity of character induction. Since M is normal in $V(\chi)$, θ vanishes off of M . Thus $V(\chi) > M \geq V(\theta)$. Suppose $I_G(\theta) = G$. By Clifford's Theorem, we have $\chi_{V(\chi)} = e\theta$. It follows that χ vanishes off of $V(\theta)$ which is properly contained in $V(\chi)$ by our above observation. This is impossible by the minimality of $V(\chi)$. Therefore $I_G(\theta) < G$. //

The proof of the following may be found in Isaacs [1, pg 82].

THEOREM 2.4: Let N be a normal subgroup of G , $\theta \in \text{Irr}(N)$ and $I = I_G(\theta)$. Let

$A = \{ \psi \in \text{Irr}(I) : [\psi_N, \theta] \neq 0 \}$, $B = \{ \chi \in \text{Irr}(G) : [\chi_N, \theta] \neq 0 \}$. Then

- i) If $\psi \in A$ then $\psi \mapsto \psi^G$ is a bijection of A onto B
- ii) If $\psi^G = \chi$ with $\psi \in A$ then ψ is the unique irreducible constituent of χ_I which lies in A and $[\psi_N, \theta] = [\chi_N, \theta]$.

3. PROOF OF THEOREM 1.1

Let G be a p -group, $\chi \in \text{Irr}(G)$, $\chi(1) \neq 1$, with N a normal subgroup of G such that $V(\chi) \leq N \leq V(\chi)\phi(G)$. Let ψ be an irreducible constituent of χ_N . Assume $I_G(\psi) = G$. We want to show that $\psi(1) = 1$.

If χ is not faithful, replace G by $G/\ker\chi$. We may do this since every character of $G/\ker\chi$ is also a character of G . Now, we prove that every irreducible constituent of $\chi_{V(\chi)}$ is linear. Let $\theta \leq \chi_{V(\chi)}$ be irreducible. Assume $I_G(\theta) < G$. Let M be a maximal subgroup of G such that $M \geq I_G(\theta)$. Since $I_G(\theta) \geq V(\chi)$ and M is maximal, it follows that $M \geq V(\chi)\phi(G) \geq N$. By Lemma 2.2 we have

$$\chi_M = \sum_{i=1}^p \beta_i, \text{ where } \beta_i \in \text{Irr}(M) \text{ are distinct.} \quad (3.1)$$

Let $G = \langle M, g \rangle$, so $\beta_j = \beta_1 g^{j-1}$ by Clifford's Theorem. Now

$$\chi_V(\chi) = (\chi_M)_V(\chi) = \sum_{i=1}^p (\beta_i)_V(\chi) = e \sum_{x \in [G: I_G(\theta)]} \theta^x \quad (3.2)$$

by (3.1) and Clifford's Theorem.

Also

$$(\beta_1)_V(\chi) = f \sum_{m \in [M: I_G(\theta)]} \theta^m. \quad (3.3)$$

Thus

$$(\beta_k)_V(\chi) = f \sum_{m \in [M: I_G(\theta)]} \theta^{mg^{k-1}}. \quad (3.4)$$

Clearly, $\{m\}$ being a transversal for $[M: I_G(\theta)]$ implies that $\{mg^{k-1}\}$ is a transversal for $[G: I_G(\theta)]$. Since, by (3.2) and (3.4),

$$e \sum_{x \in [G: I_G(\theta)]} \theta^x = \sum_{i=1}^p (\beta_i)_V(\chi) = \sum_{i=1}^p f \sum_{m \in [M: I_G(\theta)]} \theta^{mg^{i-1}} \quad (3.5)$$

we obtain $f=e$ and $(\beta_i)_V(\chi)$ and $(\beta_j)_V(\chi)$ have no common constituents for $i \neq j$. But

$I_G(\psi) = G$ so by Clifford's Theorem $\chi_N = a\psi$, yielding

$$a\psi = \chi_N = (\chi_M)_N = \sum_{i=1}^p (\beta_i)_N. \quad (3.6)$$

Thus $(\beta_i)_V(\chi) = (a/p)\psi$ all $i = 1 \dots p$ and so $(\beta_i)_V(\chi) = ((\beta_i)_N)_V(\chi) = (a/p)\psi_V(\chi)$

for all i . This is impossible since the characters $(\beta_i)_V(\chi)$ have no common constituents.

So $I_G(\theta) = G$ and, by Proposition 2.3, θ is linear. Now show that $V(\chi) = N$. Again let

$\theta \in \text{Irr}(V(\chi))$ such that $\theta \leq \chi_V(\chi)$. By the above argument θ is linear and it follows that

$\chi_V(\chi) = e\theta$ so $Z(\chi) \geq V(\chi)$, where $Z(\chi)$ denotes the center of χ . Thus $Z(\chi) = V(\chi)$ as $Z(\chi)$

is always contained in $V(\chi)$. Suppose $V(\chi) < N$. Because G is a p -group we can find B

normal in G such that $V(\chi) < B \leq N$ and $|B:V(\chi)| = p$. Thus $V(\chi) = Z(G)$, since $Z(\chi) = Z(G)$.

So B is a cyclic extension of the center of G and hence B is abelian and all of its irreducible characters are linear. Now

$$\chi_B = f \sum_{x \in [G: I_G(\alpha)]} \alpha^x \quad (3.7)$$

for some $\alpha \in \text{Irr}(B)$ where $f = [a, \chi_B]$. Since $\alpha^x(b) = \alpha(xbx^{-1}) = \alpha(b)$ for all $b \in B$ and

$x \in C_G(B)$ we obtain $C_G(B) \leq I_G(\alpha)$. Suppose $C_G(B) < I_G(\alpha)$. Then by maximality of $C_G(B)$, $I_G(\alpha) = G$. This would mean that $\chi_B = f\alpha$ and $B \leq Z(\chi)$, an obvious contradiction. Thus $I_G(\alpha) = C_G(B)$ and $I_G(\alpha)$ is maximal in G so

$$\chi_B = \sum_{i=1}^p \alpha_i, \text{ where } \alpha_i \text{ are distinct irreducible linear characters, } \alpha = \alpha_1.$$

Therefore $I_G(\alpha)$ is a maximal subgroup containing $B \geq V(\chi)$. Thus $I_G(\alpha) \geq V(\chi)\phi(G) \geq N$.

Now since $I_G(\psi) = G$ we have $\chi_N = e\psi$. Hence

$$\sum_{i=1}^p \alpha_i = \chi_B = (\chi_N)_B = e\psi_B. \quad (3.8)$$

It follows that

$$\psi_B = (f/e) \sum_{i=1}^p \alpha_i; \quad (3.9)$$

thus α is not invariant in N so $I_G(\alpha)$ does not contain N . This is a contradiction,

so $V(\chi) = N$. Since all constituents of $\chi_{V(\chi)} = \chi_N$ are linear we have $\psi(1) = 1$ as required.//

4. CHARACTERS THAT INDUCE χ

In Theorem 1.1 we considered certain subgroups of G . Now we will examine the relationship of some characters associated with these subgroups.

PROPOSITION 4.1: Let χ be a non-linear irreducible character of G . Let N be a normal subgroup of G with $N \geq V(\chi)$. Suppose θ is an irreducible constituent of χ_N , then $\theta^G = e\chi$ where $e^2 = |I_G(\theta): N|$.

PROOF: Since θ is a constituent of χ_N we have $[\theta, \chi_N] \neq 0$. By Frobenius Reciprocity, $[\chi, \theta^G] \neq 0$ thus χ is a constituent of θ^G . Suppose $\psi \in \text{Irr}(G)$ is a constituent of θ^G . Then $0 \neq [\theta^G, \psi] = [\theta, \psi_N]$, so $[\chi_N, \psi_N] \neq 0$ since θ is a constituent of both χ_N and ψ_N . Since $N \geq V(\chi)$, χ vanishes outside of N . Hence, by definition of inner product

$$|G| [\chi, \psi] = \sum_{g \in G} \chi(g)\psi(g^{-1}) = \sum_{g \in N} \chi(g)\psi(g^{-1}) = |N| [\chi_N, \psi_N] \quad (4.1)$$

Thus $[\chi, \psi] \neq 0$ and so $\chi = \psi$ since they are both irreducible. It follows that χ is the unique irreducible constituent of θ^G , so $\theta^G = e\chi$. By definition of induced character $\theta^G(1) = |G:N|\theta(1)$, so $|G:N|\theta(1) = e\chi(1)$. By Frobenius Reciprocity, $e = [\theta^G, \chi] = [\theta, \chi_N]$.

Clifford's Theorem gives

$$\chi(1) = \chi_N(1) = e \sum_{\theta \in [G: I_G(\theta)]} \theta^x(1) = e |G: I_G(\theta)| \theta(1) \quad (4.2)$$

Thus

$$|G:N|\theta(1) = e (e |G: I_G(\theta)| \theta(1)) \quad (4.3)$$

It follows that $e^2 = |I_G(\theta): N|$.

PROPOSITION 4.2: Let G be a p -group with a non-linear irreducible character χ . Let N be a normal subgroup of G such that $N \geq V(\chi)$ and let ψ be an irreducible constituent of χ_N . Let $\beta = I_G(\psi)$ and let β be an irreducible constituent of ψ^1 . Then $\psi^1 = e\beta$, $e^2 = |I: N|$ and $\beta^G = \chi$.

PROOF: By Proposition 4.1, $\psi^G = e\chi$ where $e^2 = |I: N|$. We have $0 \neq [\beta, \psi^1] = [\beta_N, \psi]$ by Frobenius Reciprocity. Now 2.4 tells us that β^G is irreducible. Also $\beta^G \leq (\psi^1)^G = e\chi$ since β is a constituent of ψ^1 , so that $\beta^G = \chi$. Again by Proposition 2.4, we have $[\beta_N, \psi] = [\chi_N, \psi] = e$. Thus $e = [\beta, \psi^1]$ by Frobenius Reciprocity and it follows that $\psi^1 \geq e\beta$ since β is irreducible. By definition of induced character, $\psi^1 = |I: N| \psi(1) = e^2 \psi(1)$, so $e^2 \geq e\beta(1)$. Since $\beta_N \geq e\psi$ we have $\beta(1) \geq e\psi(1)$. Thus $e^2 \psi(1) = \psi^1(1) \geq e\beta(1) \geq e(e\psi(1))$. It follows that $\psi^1(1) = e\beta(1)$ so $\psi^1 = e\beta$. //

Now for $\chi \in \text{Irr}(G)$, we define an *inertial decomposition series* for χ ,

denoted $[I_i, N_i, \beta_i, \psi_i]_{i=0}^m$. Here $I_0 = G = N_0$, $\beta_0 = \chi = \psi_0$, N_i is normal in I_i ,

$I_{i+1} = I_{I_i}(\psi_{i+1})$ for some $\psi_{i+1} \in \text{Irr}(N_{i+1})$, $\beta_i \in \text{Irr}(I_i)$ and $(\beta_{i+1})^{I_i} = \beta_i$. Hence we have a chain of subgroups

$$I_m \leq I_{m-1} \leq \dots \leq I_1 \leq I_0 = G$$

with associated characters $\beta_i \in \text{Irr}(I_i)$ such that $\beta_i^G = \chi$, all $i = 1 \dots m$, by transitivity of character induction.

PROPOSITION 4.3: Let G be a p -group with $\chi \in \text{Irr}(G)$. Then χ has an inertial decomposition series $[I_i, N_i, \beta_i, \psi_i]_{i=0}^m$ with $(\psi_i)^{I_i} = e_i \beta_i$ where $e_i^2 = |I_i : N_i|$ and $V(\beta_i) \leq N_{i+1} \leq V(\beta_i)\phi(I_i)$, $(\beta_i)^{N_i} = e_i \psi_i$ and $\psi_m(1) = 1$, $\psi_i \neq 1$ for $i = 1 \dots m-1$. Furthermore, $\beta_i^G = \chi$ for $i = 1 \dots m$.

PROOF: If χ is linear then it has a trivial inertial decomposition series, $[I_0, N_0, \beta_0, \psi_0]$. Assume χ is non-linear. Proof is by induction on $|G|$. Let N be a normal subgroup of G satisfying $V(\chi) \leq N \leq V(\chi)\phi(G)$. Let ψ be an irreducible constituent of χ_N

Figure 1

and let $I = I_G(\psi)$. Let β be an irreducible constituent of ψ^I . By Proposition 4.2, $\beta^G = \chi$, $\psi^I = e\beta$ where $e^2 = |I : N|$, and $\beta_N = e\psi$ by Clifford's Theorem. Set $I_0 = G = N_0$,

$\beta_0 = \chi = \psi_0$, $N_1 = N$, $l_1 = l$, $\beta_1 = \beta$, and $\psi_1 = 1$, then $[l_i, N_i, \beta_i, \psi_i]_{i=0}^1$ is an inertial decomposition series for χ as required.

Suppose $\psi(1) > 1$. Then by Theorem 1.1, $l < G$. Also $\beta_N = e\psi$ implies that $\beta(1) = e\psi(1) > 1$. Since $\beta(1) > 1$ we can apply our induction hypothesis. //

Note that, in general, we do not have l_{i+1} normal in l_i nor $N_{i+1} \leq N_i$ in an inertial decomposition series. This inertial decomposition series is illustrated in Figure 1.

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