ABSTRACT. In this paper we answer to a question posed by Marc KRANSER: It it possible to have a totally ordered noncancellative semigroup without zero divisors, and a ring hypervaluated by this semigroup? We were able to give a positive answer and provide an example.

KEY WORDS AND PHRASES. Hypervaluation, Valuation, Totally ordered semigroup, Ring.

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1. PRELIMINARIES

In what follows, all semigroups are supposed to have a unit element 1 and a zero (absorbent) element 0, such that a.0 = 0.a = 0 for all elements a in the semigroup. In any semigroup we can adjoin a zero element if it does not already have one, without changing its structure. We remark that in each semigroup 1 and 0 are unique.

DEFINITION 1. We say that a semigroup S is ordered if it is supplied with an order such that:

1. For a,b,c in S, a < bc => ca ≤ cb and ac ≤ bc.
2. 0 < 1 (hence 0 = 0.c ≤ 1.c = c for all c in S)

If the order is total S is called totally ordered.

DEFINITION 2. An hypervaluation on a ring R is a function from R onto a totally ordered semigroup S, satisfying the following conditions: For all a,b in R.

1. |a| = 0 =<=> a = 0
2. |a| = |-a|
3. |ab| = |a|.|b|
4. |a+b| ≤ Max { |a|, |b| }

Notice that if the semigroup S does not have any zero divisors then the ring R does not have any either. For if a,b ∈ R with a ≠ 0, b ≠ 0, and ab = 0, then 0 = |0| = |ab| = |a|.|b|, while |a| ≠ 0 and |b| ≠ 0. But this is impossible since S is assumed with no zero divisors. Also we easily see that a cancellative semigroup has no zero divisors, however the converse is not true in general as we shall see in what follows.
2. CONSTRUCTION OF A NON-CANCELLATIVE, TOTALLY ORDERED SEMIGROUP WITHOUT ZERO DIVISORS

We begin with an arbitrary given totally ordered semigroup \((S_1, \cdot, >)\) \(= \{0_1, a, b, \ldots\}\) where \(0_1\) its absorbent (zero) element. Consider now the set \(S_2 = S_1 \cup \{0_2\}\) that we get if we adjoin a new element \(0_2\) to the set \(S_1\). Define an operation \(\ast\) on \(S_2\) by setting
\[a \ast b = a \cdot b\] if \(a, b\) are in \(S_1\) and \(0_2 \ast a = 0_2 \ast 0_1 = 0_2\) for all \(a\) in \(S_2\). In particular \(0_2 \ast 0_1 = 0_1 \ast 0_2 = 0_2\). We observe then that:
- \((S_2, \ast)\) is a semigroup and \(0_2\) is its zero (absorbent) element (self evident).
- \((S_2, \ast)\) has no zero divisors. Indeed if \(a, b \in S_2\) with \(a \neq 0_2 b \neq 0_2\) then \(a, b \in S_1\) and by definition \(a \ast b = ab \in S_1\) and hence \(ab \neq 0_2\).
- \((S_2, \ast)\) is non-cancellative. Indeed we can take \(a, b\) in \(S_1\) with \(a \neq b\). Then \(0_1 \ast a = 0_1 a = a \neq b \neq a \ast b\), but \(0_1 = 0_1 b = 0_1 \ast 0_1\).

Finally we define a total order \(\leq\) on \(S_2\) by setting \(a \leq b\) for all \(a\) in \(S_1\), and for \(a, b, s \in S_1\), \(a \leq b\) if and only if \(a \leq b\). It is obvious that this is well defined, and that \((S_2, \leq, \ast)\) becomes a totally ordered semigroup.

3. A PROPOSITION

Notation: In what follows, we will denote by \(S_1\) an arbitrary given totally ordered semigroup, and by \(S_2\) the corresponding totally ordered non-cancellative semigroup without zero divisors, obtained from \(S_1\), by adjoining a new absorbent element \(0_2\), as it was done in section 2.

PROPOSITION: Let \(I\) be a two-sided ideal of a (not necessarily commutative) integral domain \(R\). If \(R/I\) can be hypervaluated by \(S_1\), then \(R\) can be hypervaluated by \(S_2\).

PROOF: Let \(\| \cdot \|: R/I \times S_1 = \{0_1, a, b, \ldots\}\) be a valuation from \(R/I\) onto \(S_1\). We define the function \(\|\cdot\|: R \times S_2\) by setting: For \(a\) in \(R\), \(\|a\| = 0_2\) if \(a = 0\) and \(\|a\| = \|a + I\|\) if \(a \neq 0\).

This implies that if \(a\) is in \(I\), with \(a \neq 0\), then \(\|a\| = 0_1\).

We see that \(\|\cdot\|\) thus defined, satisfies the four properties of hypervaluation:

Indeed properties (1) and (2) of definition 2 are obviously satisfied. That (3) holds for all \(a, b\) in \(R\) is immediate if at least one of them equals to zero. So we may assume \(a \neq 0, b \neq 0\), and thus \(a \neq 0, b \neq 0\) since \(R\) is an integral domain. Then \(\|ab\| = \|ab + I\| = \|(a + I)(b + I)\| = \|a + I\| \cdot \|b + I\| = \|a\| \cdot \|b\|\).

Finally (4) is also satisfied. For if \(a, b, c\) in \(R\), if at least one of them equals to zero the proof is immediate. Suppose now \(a, b \neq 0\). Then we could have \(a + b = 0\) or \(a + b \neq 0\). If \(a + b = 0\) then \(\|a + b\| = 0_2 = \|a\| \cdot \|b\|\). If \(a + b \neq 0\) then \(\|a\| = \|a + I\|\), \(\|b\| = \|b + I\|\) and \(\|a + b + \| = \|a + b + I\| = \|(a + I) + (b + I)\| \leq \Max \{|\|a + I\|, \|b + I\|\} \Max \{|\|a\|, \|b\|\}\}.

This completes the proof.

4. COFFIS THEOREM FOR HYPERVALUABILITY OF A RING

DEFINITION 3: Let \(R\) be a ring. For any element \(a\) in \(R\) we call the set of left annihilators of \(a\) to be the set \(\{x \in R : xa = 0\}\) and we denote this set by \(A_l(a)\). In an analogous way we define the set of right annihilators of \(a\) denoted by \(A_r(a)\).

THEOREM 1: (Coffi - Nikestia): Let \(R\) be a ring with a unit element \(1\). \(R\) can be hypervaluated by a totally ordered semigroup \(S\) if and only if it satisfies the following conditions:

1. For all \(a \in R\), \(A_l(a) = A_r(a)\) and we denote this set by \(A(a)\).
2. For all \(a, b \in R\), \(A(a \cdot b) = A(b \cdot a)\)
3. The class \(C = \{A(a), a \in R\}\) is totally ordered by inclusion.
In particular, $R$ possesses an hypervaluation $|\cdot|_{}$ such that $|a| \to A(a)$ is a one-to-one correspondence between $S$ and $C$.

We remark that Coffi in his construction supposes the semigroup to be commutative. The ring $R$ is not supposed to be necessarily commutative, but with an identity element $1$. The details can be found in Coffi [1]. The idea is the following: For each $a$ in $R$, its "value" $|a|$ is $A(a)$. So $|\cdot|_{} : R \to C=S$. Moreover $S$ is totally ordered by the total order defined as follows: For $a, b$ in $R$, $|a| \leq |b|$ if $A(a) \supseteq A(b)$.

5. OUR MAIN THEOREM

**THEOREM:** There exists a totally ordered non-cancellative semigroup $S$ without zero divisors, and a ring $R$ that can be hypervaluated by this semigroup.

**PROOF:** We choose an integral domain $R$ (not necessarily commutative) such that $R/I$ (for some two-sided ideal $I$ of $R$) be a ring satisfying the conditions of Coffi's theorem. Then by Coffi's theorem $R/I$ can be hypervaluated by a totally ordered semigroup $S_1$.

From $S_1$ we obtain a totally ordered, non-cancellative semigroup $S_2$ without zero divisors, as we did in section 2.

By our Proposition 1, we can hypervaluate $R$ by $S_2$ that has the desired properties. This concludes the proof of our theorem.

6. A CONCRETE EXAMPLE

We provide in this paragraph a concrete example of a ring hypervaluated by a totally ordered, non-cancellative semigroup $S_2$ without zero divisors. Let $Z$ be the ring of integers and $(16)$ the ideal in $Z$ generated by 16. It suffices to show that the ring $Z/(16)$ satisfies the conditions of Coffi's theorem and thus can be hypervaluated by a totally ordered semigroup $S_1$. Because then, by our Proposition of section 3, $Z$ can be hypervaluated by a semigroup $S_2$, having the desired properties.

Indeed since $Z/(16)$ is commutative, conditions 1 and 2 are obviously satisfied. Now if $a, b, x \in Z$ and $\overline{a}, \overline{b}, \overline{x} \in Z/(16)$ their corresponding equivalence classes, $\overline{x}$ is then an annihilator of $\overline{a}$ in $Z/(16)$ if and only if $x.a \in (16)$ i.e. iff 16 divides $x.a$. Let $(a, b)$ denote the least common multiple of two elements $a, b$ in $Z$.

Thus:

| $(a, 16)=1$ | $A(\overline{a})=|16| = |\overline{16}|$ |
| $(a, 16)=2$ | $A(\overline{a})=|8, 16|$ |
| $(a, 16)=4$ | $A(\overline{a})=|4, 8, 12, 16|$ |
| $(a, 16)=8$ | $A(\overline{a})=|2, 4, 6, 8, 10, 12, 14, 16|$ |

If in general $(a, 16)=(b, 16)$ then $A(\overline{a})=A(\overline{b})$, if $(a, 16)>(b, 16)$ then $A(\overline{a}) \supseteq A(\overline{b})$.

Condition 3 of Coffi's theorem is also therefore satisfied.

REFERENCES

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