A NOTE ON SUBORDINATION

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ABSTRACT. Suffridge showed a result for subordinate functions. The object of the present paper is to show some subordinate theorems with the aid of the result by Suffridge.

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1. INTRODUCTION.

Let \( f(z) \) and \( g(z) \) be analytic in the unit disk \( U = \{z: |z| < 1\} \). A function \( f(z) \) is said to be subordinate to \( g(z) \) if there exists a function \( \phi(z) \) analytic in the unit disk \( U \) satisfying \( \phi(0) = 0 \) and \( |\phi(z)| < 1 \) \( (z \in U) \) such that \( f(z) = g(\phi(z)) \) for \( z \in U \). We denote by \( f(z) \prec g(z) \) this relation. In particular, if \( g(z) \) is univalent in the unit disk \( U \) the subordination is equivalent to \( f(0) = g(0) \) and range \( f(z) \subset \text{range} \ g(z) \).

This concept of subordination can be traced to Lindelöf [1], but Littlewood [2],[3] and Rogosinski [4], [5] introduced the term and discovered the basic properties. Recently Suffridge [6] and Hallenbeck and Ruscheweyh [7] studied the subordinate functions and showed many interesting results for subordinations.

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the unit disk \( U \). Further let \( S \) be the subclass of \( A \) consisting of analytic and univalent functions in the unit disk \( U \). Then a function \( f(z) \) of \( S \) is said to be starlike of order \( \alpha \) if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)
\]

for some \( \alpha \) \( (0 \leq \alpha < 1) \). We denote by \( S^*(\alpha) \) the class of all starlike functions of order \( \alpha \). Further a function \( f(z) \) of \( S \) is said to be convex of order \( \alpha \) if and only if

\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)
\]

for some \( \alpha \) \( (0 \leq \alpha < 1) \). And we denote by \( K(\alpha) \) the class of convex functions of
order $\alpha$. It is well-known that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, $S^*(\alpha) \subset S^*$, $K(\alpha) \subset K$, and $S^*(0) \equiv S^*$, $K(0) \equiv K$ for $\alpha = 0$.

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [8], and latter studied by Schild [9], MacGregor [10] and Pinchuk [11]. Further, recently, some classes defined by using the extremal function $z/(1-z)^2(1-\alpha)$ for $S^*(\alpha)$ were studied by Ruscheweyh [12], Sheil-Small, Silverman and Silvia [13], Silverman and Silvia [14], and Ahuja and Silverman [15].

Our main tool in this paper is the following result by Suffridge [15].

**LEMMA.** Let the function $f(z) = \sum_{n=2}^\infty a_n z^n$ be analytic in the unit disk $U$ and the function $g(z)$ be in the class $S^*$. If $f(z)$ is subordinate to $g(z)$, that is,

\[
zf'(z) \preceq zg'(z)
\]

for $z \in U(r) = \{z : |z| \leq r, 0 \leq r < 1\}$.

2. **SUBORDINATION THEOREMS.**

In this section, we show some subordination theorems with the aid of Lemma.

**THEOREM 1.** Let the function $f(z)$ defined by (1.1) be in the class of $K(\alpha)$.

Then $f'(re^{i\theta}) (0 \leq r < 1)$ is contained in the image domain of the closed disk $U(r)$ under the function $e^{4(\alpha-1)/(1-z)}$. Further it lies for $r \neq 0$ on the boundary of this image domain if and only if

\[
f(z) = \int_0^z e^{4(1-\alpha)/(1-ct)} dt.
\]

(2.1)

where $|c| = 1$.

**PROOF.** Since $f(z)$ is in the class $K(\alpha)$, $f(z)$ satisfies that

\[
Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha - 1 (z \in U).
\]

Note that $zf''(z)/f'(z) = 2a_2z + \ldots$ is analytic in the unit disk $U$, and that the function $z/(1-z)^2$ is starlike with respect to the origin and $Re \{z/(1-z)^2\} > -1/4$. Hence we have that

\[
\frac{zf''(z)}{f'(z)} \preceq \frac{4(1-\alpha)z}{(1-z)^2} (z \in U).
\]

Consequently, by using Lemma, it follows that $\log f'(re^{i\theta})$ is contained in the image domain of $U(r)$ under the function $e^{4(\alpha-1)/(1-z)}$, where log is understood to be that branch which vanishes at the point one. Thus we can see that $f'(re^{i\theta})$ lies for $r \neq 0$ on the boundary of the image domain of $U(r)$ under $e^{4(\alpha-1)/(1-z)}$. Further $f'(re^{i\theta})$ lies for $r \neq 0$ on the boundary of the image domain $U(r)$ under $e^{4(\alpha-1)/(1-z)}$ if and only if

\[
\frac{zf''(z)}{f'(z)} = \frac{4(1-\alpha)z}{(1-cz)^2} (|c| = 1),
\]

hence further, $f(z)$ is the function of the form (2.1). This completes the proof of the theorem.
THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha)$.
Then $f(re^{i\theta})/re^{i\theta}$ $(0 \leq r < 1)$ is contained in the image domain of the closed disk $u(r)$ under the function $e^{4(-1)/(1-z)}$. Further it lies for $r \neq 0$ on the boundary of this image domain if and only if $f(z) = ze^{4(1-\alpha)/(1-ez)}$, where $|e| = 1$.

PROOF. Since $f(z) \in S^*(\alpha)$, $f(z)$ satisfies that
\[
\text{Re}\left\{\frac{zf'(z)}{f(z)} - 1\right\} > \alpha - 1 \quad (z \in u)
\]
and the function $zf'(z)/f(z) - 1 = a_2 z + \ldots$ is analytic in the unit disk $u$. Hence $zf'(z)/f(z) - 1$ takes values in the image domain of the unit disk $u$ under the function $4(1-\alpha)z/(1-z)^2$, that is,
\[
zf'(z)/f(z) - 1 < \frac{4(1-\alpha)z}{(1-z)^2} \quad (z \in u).
\]
By virtue of Lemma, we observe that $\log f(re^{i\theta})/re^{i\theta}$ $(0 \leq r < 1)$ is contained in the image domain of $u(r)$ under $4(\alpha-1)/(1-z)$ and it lies for $r \neq 0$ on the boundary of this image domain if and only if
\[
zf'(z)/f(z) - 1 = \frac{4(1-\alpha)ez}{(1-ez)^2} \quad (|e| = 1),
\]
hence further, $f(z) = ze^{4(1-\alpha)/(1-ez)}$. This gives the result we require.

Finally we show a theorem for functions $f(z)$ satisfying $\text{Re}\{zf'(z)\} > \alpha (\alpha > 0)$.

THEOREM 3. Let the function $f(z)$ defined by (1.1) satisfy $\text{Re}\{zf'(z)\} > \alpha (\alpha > 0)$.
Then $f(re^{i\theta})$ $(0 \leq r < 1)$ is contained in the image domain of the closed disk $u(r)$ under the function $-4a/(1-z)$. Further it lies for $r \neq 0$ on the boundary of this domain if and only if $f(z) = -4a/(1-ez)$, where $|e| = 1$.

PROOF. We note that the function $zf'(z) = z + 2a_2 z^2 + \ldots$ takes values in the image domain of the unit disk $u$ under the function $-4az/(1-z)^2$ which belongs to the class $S^*$. Therefore we can prove the theorem by using the same technique as in the one of Theorem 1 with the aid of Lemma.

REFERENCES