ABSTRACT. A subset \( N \) of a topological space is defined to be a \( \theta \)-neighborhood of \( x \) if there exists an open set \( U \) such that \( x \in U \subseteq \text{Cl} U \subseteq N \). This concept is used to characterize the following types of functions: weakly continuous, \( \theta \)-continuous, strongly \( \theta \)-continuous, almost strongly \( \theta \)-continuous, weakly \( \delta \)-continuous, weakly open and almost open functions. Additional characterizations are given for weakly \( \delta \)-continuous functions. The concept of \( \theta \)-neighborhood is also used to define the following types of open maps: \( \theta \)-open, strongly \( \theta \)-open, almost strongly \( \theta \)-open, and weakly \( \delta \)-open functions.

KEY WORDS AND PHRASES. \( \theta \)-neighborhood, weakly continuous function, \( \theta \)-continuous function, strongly \( \theta \)-continuous function, almost strongly \( \theta \)-continuous function, weakly \( \delta \)-continuous function, weakly open function, almost open function, \( \theta \)-open function, strongly \( \theta \)-open function, almost strongly \( \theta \)-open function, weakly \( \delta \)-open function.

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1. INTRODUCTION.

Near-continuity has been investigated by many authors including Levine [1], Long and Herrington [2], Noiri [3], and Rose [4]. Near-openness has been developed by Rose [5] and Singal and Singal [6]. The purpose of this note is to characterize several types of near-continuity and near-openness in terms of the concept of \( \theta \)-neighborhood. These characterizations clarify both the nature of these functions and the relationships among them. Additional characterizations of weak \( \delta \)-continuity are given. The concept of \( \theta \)-neighborhood also leads to the definition of several new types of near-open functions.

2. DEFINITIONS AND NOTATION.

The symbols \( X \) and \( Y \) denote topological spaces with no separation axioms assumed unless explicitly stated. Let \( U \) be a subset of a space \( X \). The closure of \( U \) and the interior of \( U \) are denoted by \( \text{Cl} U \) and \( \text{Int} U \) respectively. The set \( U \) is said to be regular open (regular closed) if \( U = \text{Int} \text{Cl} U \) (\( U = \text{Cl} \text{Int} U \)). The \( \theta \)-closure (\( \delta \)-closure) (Velicko [7]) of \( U \) is the set of all \( x \) in \( X \) such that every closed neighborhood (the interior of every closed neighborhood) of \( x \) intersects
U. The $\theta$-closure and the $\delta$-closure of $U$ are denoted by $\text{Cl}_\theta U$ and $\text{Cl}_\delta U$ respectively. The set $U$ is called $\theta$-closed ($\delta$-closed) if $U = \text{Cl}_\theta U$ ($U = \text{Cl}_\delta U$). A set is said to be $\theta$-open ($\delta$-open) if its complement is $\theta$-closed ($\delta$-closed). For a given space $X$ the collection of all $\theta$-open sets and the collection of all $\delta$-open sets both form topologies. The space $X$ with the $\theta$-open ($\delta$-open) topology will be signified by $X_\theta$ ($X_\delta$).

**DEFINITION 1.** A function $f: X \to Y$ is said to be weakly continuous (Levine [1]) ($\theta$-continuous (Fomin [8]), strongly $\theta$-continuous (Long and Herrington [2]), almost strongly $\theta$-continuous (Noiri and Kang [9]), weakly $\delta$-continuous (Baker [10]) if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists an open neighborhood $U$ of $x$ such that $f(U) \subseteq \text{Cl}_\theta V$ ($f(\text{Cl}_\delta U) \subseteq \text{Cl}_\theta V$, $f(\text{Cl}_\delta U) \subseteq V$, $f(\text{Cl}_\delta U) \subseteq \text{Int Cl}_\delta V$, $f(\text{Int Cl}_\delta U) \subseteq \text{Cl}_\delta V$).

**DEFINITION 2.** A function $f: X \to Y$ is said to be weakly open (Rose [5]) (almost open (Rose [5])) provided that for each open subset $U$ of $X$, $f(U) \subseteq \text{Int f(Cl}_\delta U$ ($f(U) \subseteq \text{Int Cl}_\theta f(U)$).

**DEFINITION 3.** A subset $N$ of a space $X$ is said to be a $\theta$-neighborhood ($\delta$-neighborhood) of a point $x$ in $X$ if there exists an open set $U$ such that $x \in U \subseteq \text{Cl}_\theta U \subseteq N$ ($x \in U \subseteq \text{Int Cl}_\theta U \subseteq N$).

Note that a $\theta$-neighborhood is not necessarily a neighborhood in the $\theta$-topology, but a $\delta$-neighborhood is a neighborhood in the $\delta$-topology.

### 3. NEAR-CONTINUOUS FUNCTIONS.

The main results can be paraphrased as follows: weak continuity corresponds to "$f^{-1}$ ($\theta$-neighborhood) = neighborhood"; $\theta$-continuity corresponds to "$f^{-1}$ ($\theta$-neighborhood) = $\theta$-neighborhood"; strong $\theta$-continuity corresponds to "$f^{-1}$ (neighborhood) = $\theta$-neighborhood"; almost strong $\theta$-continuity corresponds to "$f^{-1}$ ($\delta$-neighborhood) = $\theta$-neighborhood"; and weak $\delta$-continuity corresponds to "$f^{-1}$ ($\theta$-neighborhood) = $\delta$-neighborhood".

**THEOREM 1.** A function $f: X \to Y$ is weakly continuous if and only if for each $x$ in $X$ and each $\theta$-neighborhood $N$ of $f(x)$, $f^{-1}(N)$ is a neighborhood of $x$.

**PROOF.** Assume $f$ is weakly continuous. Let $x \in X$ and let $N$ be a $\theta$-neighborhood of $f(x)$. Then there exists an open set $V$ such that $f(x) \in V \subseteq \text{Cl}_\theta V \subseteq N$. Since $f$ is weakly continuous, there exists an open neighborhood $U$ of $x$ such that $f(U) \subseteq \text{Cl}_\theta V \subseteq N$. Thus $x \in U \subseteq f^{-1}(N)$ and hence $f^{-1}(N)$ is a neighborhood of $x$.

Assume for each $x \in X$ and each $\theta$-neighborhood $N$ of $x$ that $f^{-1}(N)$ is a neighborhood of $x$. Let $x \in X$ and let $V$ be an open neighborhood of $f(x)$. Since $\text{Cl}_\theta V$ is a $\theta$-neighborhood of $f(x)$, $f^{-1}(\text{Cl}_\theta V)$ is a neighborhood of $x$. Thus there is an open set $U$ for which $x \in U \subseteq f^{-1}(\text{Cl}_\theta V)$ and $f(U) \subseteq \text{Cl}_\theta V$ which proves $f$ is weakly continuous.

**THEOREM 2.** A function $f: X \to Y$ is $\theta$-continuous if and only if for each $x$ in $X$ and each $\theta$-neighborhood $N$ of $f(x)$, $f^{-1}(N)$ is a $\theta$-neighborhood of $x$. 

CHARACTERIZATIONS OF SOME NEAR-CONTINUOUS FUNCTIONS

PROOF. Assume \( f: X \rightarrow Y \) is \( \theta \)-continuous. Let \( x \in X \) and let \( N \) be a \( \theta \)-neighborhood of \( f(x) \). Then there exists an open set \( V \) for which \( f(x) \in V \subseteq \text{Cl}V \subseteq N \). By the \( \theta \)-continuity of \( f \), there exists an open neighborhood \( U \) of \( x \) such that \( f(\text{Cl} U) \subseteq \text{Cl} V \subseteq N \). Thus \( x \in U \subseteq \text{Cl} U \subseteq f^{-1}(N) \) and hence \( f^{-1}(N) \) is a \( \theta \)-neighborhood of \( x \).

Assume for each \( x \) in \( X \) and for each \( \theta \)-neighborhood \( N \) of \( f(x) \) that \( f^{-1}(N) \) is a \( \theta \)-neighborhood of \( x \). Let \( x \in X \) and let \( V \) be an open neighborhood of \( f(x) \). Since \( \text{Cl} V \) is a \( \theta \)-neighborhood of \( f(x) \), \( f^{-1}(\text{Cl} V) \) is a \( \theta \)-neighborhood of \( x \). Hence there exists an open set \( U \) for which \( x \in U \subseteq \text{Cl} U \subseteq f^{-1}(\text{Cl} V) \). That is, \( f(\text{Cl} U) \subseteq \text{Cl} V \) and thus \( f \) is \( \theta \)-continuous.

The proof of the following theorem is similar to that of Theorem 2 and is omitted.

THEOREM 3. A function \( f: X \rightarrow Y \) is strongly \( \theta \)-continuous if and only if for each \( x \) in \( X \) and each \( \theta \)-neighborhood \( N \) of \( f(x) \), \( f^{-1}(N) \) is a \( \theta \)-neighborhood of \( x \).

THEOREM 4. A function \( f: X \rightarrow Y \) is almost strongly \( \theta \)-continuous if and only if for each \( x \) in \( X \) and each \( \delta \)-neighborhood \( N \) of \( f(x) \), \( f^{-1}(N) \) is a \( \theta \)-neighborhood of \( x \).

PROOF. Assume \( f: X \rightarrow Y \) is almost strongly \( \theta \)-continuous. Let \( x \in X \) and let \( N \) be a \( \delta \)-neighborhood of \( f(x) \). Then there exists an open set \( V \) such that \( f(x) \in V \subseteq \text{Int} \text{Cl} V \subseteq N \). Since \( f \) is almost strongly \( \theta \)-continuous, there exists an open neighborhood \( U \) of \( x \) for which \( f(\text{Cl} U) \subseteq \text{Int} \text{Cl} V \subseteq N \). Then \( x \in U \subseteq \text{Cl} U \subseteq f^{-1}(\text{Cl} V) \) which proves that \( f^{-1}(N) \) is a \( \theta \)-neighborhood of \( x \).

Assume for each \( x \in X \) and each \( \delta \)-neighborhood \( N \) of \( f(x) \) that \( f^{-1}(N) \) is a \( \theta \)-neighborhood of \( x \). Let \( x \in X \) and let \( V \) be an open neighborhood of \( f(x) \). Since \( \text{Int} \text{Cl} V \) is a \( \delta \)-neighborhood of \( f(x) \), \( f^{-1}(\text{Int} \text{Cl} V) \) is a \( \theta \)-neighborhood of \( x \). Hence there is an open set \( U \) such that \( x \in U \subseteq \text{Cl} U \subseteq f^{-1}(\text{Int} \text{Cl} V) \). That is, \( f(\text{Cl} U) \subseteq \text{Int} \text{Cl} V \) and hence \( f \) is almost strongly \( \theta \)-continuous.

THEOREM 5. A function \( f: X \rightarrow Y \) is weakly \( \delta \)-continuous if and only if for each \( x \in X \) and each \( \theta \)-neighborhood \( N \) of \( f(x) \), \( f^{-1}(N) \) is a \( \delta \)-neighborhood of \( x \).

The proof of this theorem is similar to that of Theorem 4. The following theorem gives additional characterizations of weak \( \delta \)-continuity. These results are analogous to those obtained by Noiri and Kang in [9] for almost strongly \( \theta \)-continuous functions.

LEMMA. Let \( X \) be a space and \( H \subseteq X \). Then
(a) \( \text{Cl}_\delta H = \{ x \in X : \text{every } \theta \text{-neighborhood of } x \text{ intersects } H \} \) and
(b) \( \text{Cl}_\delta H = \{ x \in X : \text{every } \delta \text{-neighborhood of } x \text{ intersects } H \} \).

The proof follows easily from the definitions.

THEOREM 6. For \( f: X \rightarrow Y \) the following statements are equivalent:
(a) \( f: X \rightarrow Y \) is weakly \( \delta \)-continuous.
(b) For each \( H \subseteq X \), \( f(\text{Cl}_\delta H) \subseteq \text{Cl}_\delta f(H) \).
(c) For each \( K \subseteq Y \), \( \text{Cl}_\delta f^{-1}(K) \subseteq f^{-1}(\text{Cl}_\delta K) \).
(d) \( f: X \rightarrow Y \) is weakly continuous.
PROOF. (a) => (b). Let $H \subseteq X$ and let $y \in f(Cl_{\delta} H)$. Then there exists an $x$ in $Cl_{\delta} H$ such that $y = f(x)$. Let $N$ be a $\theta$-neighborhood of $f(x)$. By Theorem 5 $f^{-1}(N)$ is a $\delta$-neighborhood of $x$. Since $x \in Cl_{\delta} H$, $f^{-1}(N) \cap H \neq \emptyset$. That is, $N \cap f(H) \neq \emptyset$. Hence $y \in Cl_{\theta} f(H)$. Thus $f(Cl_{\delta} H) \subseteq Cl_{\theta} f(H)$.

(b) => (c). Let $K \subseteq Y$. By (b) $f(Cl_{\delta} f^{-1}(K)) \subseteq Cl_{\theta} f(f^{-1}(K)) \subseteq Cl_{\theta} K$. Thus $Cl_{\delta} f^{-1}(K) \subseteq f^{-1}(Cl_{\theta} K)$.

(c) => (d). Let $x \in X$ and let $V$ be an open neighborhood of $f(x)$. Since $Cl V$ is a $\theta$-neighborhood of $f(x)$, $f(x) \notin Cl_{\theta} (Y - Cl V)$. Hence $f^{-1}(Cl_{\theta} (Y - Cl V)) \subseteq \emptyset$. By (c) $x \notin Cl_{\delta} f^{-1} (Y - Cl V)$. Thus there is a neighborhood $U$ of $x$ such that $(Int \ Cl U) \cap f^{-1} (Y - Cl V) = \emptyset$. Then $f(Int \ Cl U) \subseteq Cl V$. Since $Int \ Cl U$ is a regular open, $f: X_{\delta} \rightarrow Y$ is weakly continuous.

(d) => (a). Let $x \in X$ and let $V$ be an open neighborhood of $f(x)$. Since $f: X_{\delta} \rightarrow Y$ is weakly continuous, there exists a $\delta$-open set $W$ containing $x$ such that $f(W) \subseteq Cl V$. Then there is a regular open set $U$ for which $x \in U \subseteq W$. Then $f(Int \ Cl U) = f(U) \subseteq f(W) \subseteq Cl V$ and hence $f$ is weakly $\delta$-continuous.

4. NEAR-OPEN FUNCTIONS.

In this section weak openness and almost openness are characterized in terms of the concept of $\theta$-neighborhood.

THEOREM 7. A function $f: X \rightarrow Y$ is weakly open if and only if for each $x \in X$ and each $\theta$-neighborhood $N$ of $x$, $f(N)$ is a neighborhood of $f(x)$.

PROOF. Assume $f$ is weakly open. Let $x \in X$ and let $N$ be a $\theta$-neighborhood of $x$. Then there is an open set $U$ such that $x \in U \subseteq Cl U \subseteq N$. Since $f$ is weakly open $f(x) \in f(U) \subseteq Int f(Cl U) \subseteq Int f(N)$. Hence $f(N)$ is a neighborhood of $f(x)$.

Assume for each $x$ in $X$ and each $\theta$-neighborhood $N$ of $x$ that $f(N)$ is a neighborhood of $f(x)$. Let $U$ be an open set in $X$. Suppose $x \in U$. Since $Cl U$ is a $\theta$-neighborhood of $x$, $f(Cl U)$ is a neighborhood of $f(x)$. Hence $f(x) \in Int f(Cl U)$. Thus $f(U) \subseteq Int f(Cl U)$ and $f$ is weakly open.

The proof of the following theorem is similar and is omitted.

THEOREM 8. A function $f: X \rightarrow Y$ is almost open if and only if for each $x \in X$ and each neighborhood $N$ of $x$, $Cl f(N)$ is a $\theta$-neighborhood of $f(x)$.

Theorem 7 and the characterizations of near-continuous functions in Section 3 suggest the following definitions of near-open functions.

DEFINITION 4. A function $f: X \rightarrow Y$ is said to be $\theta$-open (strongly $\theta$-open, almost strongly $\theta$-open, weakly $\delta$-open) if for each $x \in X$ and each $\theta$-neighborhood (neighborhood, $\delta$-neighborhood, $\theta$-neighborhood) $N$ of $x$, $f(N)$ is a $\theta$-neighborhood ($\theta$-neighborhood, $\theta$-neighborhood, $\delta$-neighborhood) of $f(x)$.

The following theorems characterize these near-open functions in terms of the closure and interior operators. Since the proofs are all similar, only the first theorem is proved.

THEOREM 9. A function $f: X \rightarrow Y$ is $\theta$-open if and only if for each $x \in X$ and each open neighborhood $U$ of $x$, there exists an open neighborhood $V$ of $f(x)$ such that $Cl V \subseteq f(Cl U)$. 
PROOF. Assume \( f: X \to Y \) is 0-open. Let \( x \in X \) and let \( U \) be an open neighborhood of \( x \). Since \( f(Cl U) \) is a 0-neighborhood of \( f(x) \), there exists an open set \( V \) such that \( f(x) \in V \subseteq Cl V \subseteq f(Cl U) \).

Assume that for each \( x \in X \) and each open neighborhood \( U \) of \( x \) there exists an open neighborhood \( V \) of \( f(x) \) for which \( Cl V \subseteq f(Cl U) \). Let \( x \in X \) and let \( N \) be a 0-neighborhood of \( x \). Then there is an open set \( U \) for which \( x \in U \subseteq Cl U \subseteq N \). There exists an open set \( V \) such that \( f(x) \in V \subseteq Cl V \subseteq f(Cl U) \subseteq f(N) \). Hence \( f(N) \) is a 0-neighborhood of \( f(x) \) and \( f \) is 0-open.

THEOREM 10. A function \( f: X \to Y \) is strongly 0-open if and only if for each \( x \in X \) and each open neighborhood \( U \) of \( x \), there exists an open neighborhood \( V \) of \( f(x) \) such that \( Cl V \subseteq f(U) \).

THEOREM 11. A function \( f: X \to Y \) is almost strongly 0-open if and only if for each \( x \in X \) and each open neighborhood \( U \) of \( x \) there exists an open neighborhood \( V \) of \( f(x) \) such that \( Cl V \subseteq f(Int Cl U) \).

THEOREM 12. A function \( f: X \to Y \) is weakly 0-open if and only if for each \( x \in X \) and each open neighborhood \( U \) of \( x \), there exists an open neighborhood \( V \) of \( f(x) \) such that \( Int Cl V \subseteq f(Cl U) \).

We have the following implications: almost open \( \implies \) st. 0-open \( \implies \) almost st. 0-open \( \implies \) weak 0-open \( \implies \) weak open. The following examples show that these implications are not reversible.

EXAMPLE 1. Let \( X = \{a, b\} \), \( T_1 = \{X, \emptyset, \{a\}\} \), \( Y = \{a, b, c\} \), and \( T_2 = \{Y, \emptyset, \{a\}, \{a, b\}\} \). The inclusion mapping: \( (X, T_1) \to (Y, T_2) \) is weak open but not weak 0-open.

In the next example the space \( (Y, T_2) \) is from Example 2.2 in Noiri and Kang [9].

EXAMPLE 2. Let \( (X, T_1) \) be as in Example 1. Let \( Y = \{a, b, c, d\} \), and \( T_2 = \{Y, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). The inclusion mapping: \( (X, T_1) \to (Y, T_2) \) is weak 0-open but not 0-open.

EXAMPLE 3. Let \( (Y, T_2) \) be as in Example 2. The identity mapping: \( (Y, T_2) \to (Y, T_2) \) is 0-open but not almost strongly 0-open.

EXAMPLE 4. Let \( X = \{a, b, c\}, T_1 = \{X, \emptyset, \{a\}, \{a, c\}\} \) and \( T_2 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}\} \). The identity mapping: \( (X, T_1) \to (X, T_2) \) is almost strongly 0-open and almost open, but not strongly 0-open.

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