THE PACKING AND COVERING OF THE COMPLETE GRAPH I: THE FORESTS OF ORDER FIVE

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ABSTRACT. The maximum number of pairwise edge disjoint forests of order five in the complete graph K_n , and the minimum number of forests of order five whose union is K_n , are determined.

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1. INTRODUCTION

Graphs in this paper are finite with no multiple edges or loops. Beineke [1] defined the general covering (respectively, packing) problem as follows:

For a given graph G find the minimum (maximum) number of edge disjoint subgraphs of G such that each subgraph has a specified property P and the union of the subgraphs is G .

Solutions of these problems are known only for a few properties P , when G is arbitrary. In most cases G is taken to be the complete graph K_n or the complete bipartite graph $K_{m,n}$ (for particular references one may look at Roditty [2]).

DEFINITION: The complete graph K_n is said to have a G-decomposition if it is the union of edge disjoint subgraphs each isomorphic to G . We denote such a decomposition by $G|K_n$.

The G-decomposition problem is to determine the set $N(G)$ of natural numbers such that K_n has a G-decomposition if and only if $n \in N(G)$. Note that G-decomposition is actually an exact packing and covering. In the proof of our problems of packing and covering, we make great use of the results obtained for the G-decomposition problem in cases when G has five vertices. As usual $[x]$ will denote the largest integer not exceeding x and $\{x\}$ the least integer not less than x . We will let $e(G)$ denote the number of edges of the graph G and $H = \bigcup_{i=1}^t G_i$ will show that the graph H is the union of t edge disjoint graphs G_i , $i=1,2,\dots,t$.

The Theorem of this paper solves variations of the covering and packing problems for the four graphs below:

(i) F_1 : $\begin{array}{cccccc} x & y & z & u & v \\ \circ & \text{---} & \circ & \text{---} & \circ \end{array}$ denoted $[(x,y,z)(u,v)]$

(ii) F_2 : $\begin{array}{cccccc} x & y & z & u & v \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}$ denoted $[(x,y,z,u,v)]$

Our theorem may now be stated as THEOREM (Packing and Covering).

Let F be F_1, F_2 or F_3 and $n \geq 5$ or F be F_4 and $n \geq 7$ then

(i) The maximum number of edge disjoint graphs F which are subgraphs of the complete graph K_n is

$$\lfloor e(K_n)/e(F) \rfloor.$$

(ii) The minimum number of graphs F whose union is the complete graph K_n is

$$\lceil e(K_n)/e(F) \rceil.$$

2. PROOF OF THE THEOREM

We give a separate proof for each choice of F .

F_1 : Proving the Theorem true for $n \geq 5$ is a straightforward exercise. Bermond et al. [3] show that

$$N(F_1) = \{n \mid n \equiv 0, 1 \pmod{3}, n \geq 6\}. \quad (2.1)$$

Thus we have to consider only $n = 3m + 2, m \geq 2$. Observe that

$$K_{3m+2} = K_{3m} \cup K_{2,3m} \cup K_2, \quad m \geq 2. \quad (2.2)$$

by (2.1) K_{3m} has an F_1 -decomposition. Since $K_{2,3m} = mK_{2,3}$ and $K_{2,3}$ can be decomposed easily into two graphs F_1 , it follows that $K_{2,3m}$ has an F_1 -decomposition. Only K_2 in (2.2) is left non-packed. Hence, the Theorem is proved in this case.

F_2 : The proof will examine several cases depending on the value of n . The following table summarizes the cases $n = 5, 6, 7, 8m$, and $8m + 1$ for $m \geq 1$.

n	packing	remains for covering
5	$(0,1,2,3,4); (1,3,0,4,2)$	$(0,2); (1,4)$
6	$(0,1,2,3,4); (0,5,4,1,3); (0,4,2,5,3)$	$(0,3), (0,2), (1,5)$
7	$(0,1,2,3,4); (0,2,4,6,1); (1,3,5,0,4)$ $(1,4,5,6,0); (1,5,2,6,3)$	$(0,3)$
$8m, 8m+1$	F_2 - decomposition [4]	

Table 1

We still have to prove the theorem for the cases:

$$n = 8m + k, \quad k = 2, \dots, 7$$

$k = 2$.

Let

$$K_{8m+2} = K_{8m} \cup K_{2,8m} \cup K_2. \quad (2.3)$$

The graph K_{8m} has an F_2 -decomposition. since $K_{2,8m} = 2mK_{2,4}$ and $K_{2,4}$ can be decomposed easily into two graphs F_2 , it follows that $K_{2,8m}$ has an F_2 -decomposition. Only K_2 in (2.3) is left non-packed.

$k = 3$.

Let

$$K_{8m+3} = K_{8m+1} \cup K_{2,8m+1} \cup K_2. \quad (2.4)$$

The graph K_{8m+1} has an F_2 -decomposition $K_{2,8m+1} = K_{2,8m} \cup K_{2,1}$ and $K_{2,8m}$ has an F_2 -decomposition as we saw above. This decomposition of $K_{2,8m}$ can be done in such a way that the edge $(8m-1, 8m+2)$ is at one end of the F_2 which includes it and the point $8m-1$ is an end-point of that F_2 . Thus we can replace the edge $(8m-1, 8m+2)$ with the edge $(8m, 8m+2)$. Only the edges $(8m, 8m+1)$, $(8m+1, 8m+2)$, $(8m-1, 8m+2)$ now remain non-packed, and they can be included in one more F_2 .

$k = 4$.

Note that

$$K_{8m+4} = K_{8m} \cup K_{4,8m} \cup K_4. \quad (2.5)$$

The graph K_{8m} has an F_2 -decomposition. Now

$$K_{4,8m} \cup K_4 = 2(2m-1)K_{2,4} \cup 2K_{2,4} \cup K_4 \quad (2.6)$$

and the $2K_{2,4}$'s can be selected to be vertex disjoint. Since $K_{2,4}$ has an F_2 -decomposition, so does $2(2m-1)K_{2,4}$. We need only to show that $2K_{2,4} \cup K_4$ can be packed by 5 F_2 graphs, leaving two non-packed edges.

Let $V(2K_{2,4}) = \{1, 2, \dots, 8, a, b, c, d\}$, $V(K_4) = \{a, b, c, d\}$.

Then, the 5 graphs of the packing of $2K_{2,4} \cup K_4$ are:

$$(a, 3, b, c, 8); (1, a, d, 7, c); (4, b, d, c, 5); (a, c, 6, d, 8); (1, b, 2, a, 4).$$

The edges $(d, 5)$ and (a, b) are left non-packed.

$k = 5$.

Let

$$K_{8m+5} = K_{8m+1} \cup K_{4,8m} \cup K_{4,1} \cup K_4. \quad (2.7)$$

The graph K_{8m+1} has an F_2 -decomposition. In the case $k = 4$ we saw that $K_{4,8m} \cup K_4$ has an F_2 -packing leaving two non-packed edges.

Let $V(K_4) = \{a, b, c, d\}$ and $V(K_{8m+1}) = Z_{8m+1}$. Denote the non-packed edges by (a, b) and $(8m-1, d)$. We show that $G = K_{4,1} \cup \{(a, b), (8m-1, d)\}$ has an F_2 -packing leaving two non-packed edges. The F_2 of this packing is $(8m-1, d, 8m, a, b)$. The non-packed edges are: $(c, 8m)$ and $(8m, b)$.

$k = 6$.

Write

$$K_{8m+6} = K_{8m} \cup K_{6,8m} \cup K_6.$$

The graph K_{8m} has an F_2 -decomposition. Observe that $K_{6,8m} = 3K_{2,8m}$. In the case $k = 2$ we saw that $F_2 | K_{2,8m}$. Table 1 shows that K_6 has F_2 -packing leaving three non-packed edges as required, and these three can be included in one more F_2 .

$k = 7$.

Let

$$K_{8m+7} = K_{8m+1} \cup K_{6,8m} \cup K_7.$$

The graph K_{8m+1} has an F_2 -decomposition, and $F_2 | K_{6,8m}$, as was shown above. By Table 1 we know that the graph K_7 has an F_2 -packing leaving one non-packed edge. The Theorem has now been proved for F_2 since all cases have been considered.

F_3 : The proof will consider the same cases as the proof for F_2 .

n	packing	remains for covering
5	$(0,1,2;3,4);(1,4,0;2,3)$	$(1,3),(3,4)$
6	$(0,1,2;3,4);(3,4,5;0,2);(0,3,1;4,5)$	$(0,2),(0,4),(3,5)$
7	$(3,2,0;1,6);(5,4,1;2,3);(1,6,3;4,5)$ $(2,4,0;3,5);(1,5,6;2,4)$	$(2,5)$
$8m, 8m+1$	F_3 - decomposition [4]	

Table 2

We now have to prove the theorem for the cases:

$$n = 8m+k, \quad k = 2, \dots, 7, \quad m \geq 1.$$

$k = 2$.

Let K_{8m+2} be as in (2.3). The graph K_{8m} has an F_3 -decomposition. Since $K_{2,8m} = 2mK_{2,4}$ and $K_{2,4}$ can be decomposed easily into two F_3 graphs, it follows that $K_{2,8m}$ has an F_3 -decomposition. Only K_2 in (2.3) is left non-packed. Hence, the Theorem is proved in this case.

$k = 3$.

Let K_{8m+3} be as in (2.4). K_{8m+1} has an F_3 -decomposition. $K_{2,8m+1} = K_{2,8m} \cup K_{2,1}$. The graph $K_{2,8m}$ has an F_3 -decomposition as was shown above. Replace the edge $(8m-4, 8m+2)$ which appears in some F_3 in the decomposition of $K_{2,8m}$, with the edge $(8m, 8m+2)$. Then the edges $(8m-4, 8m+2)$, $(8m+2, 8m+1)$, $(8m+1, 8m)$ remain non-packed, but could be included in one additional F_3 .

$k = 4$.

Let K_{8m+4} be as in (2.5). The graph K_{8m} has an F_3 -decomposition. Let, $K_{4,8m} \cup K_4$ be as in (2.6). Since $K_{2,4}$ has an F_3 -decomposition, so does $2(2m-1)K_{2,4}$. We show that $2K_{2,4} \cup K_4$ can be packed by five F_3 graphs, leaving two non-packed edges.

Let $V(2K_{2,4}) = \{1, 2, \dots, 8, a, b, c, d\}$ and $V(K_4) = \{a, b, c, d\}$.
Then the five graphs F_3 are:

$$(c, 6, d; 7, 8), (d, 5, c; 7, 8), (4, a, c; d, b), (1, a, b; 4, d), (a, 3, b; 1, 2).$$

The edges $(a, 2)$ and (a, d) are left non-packed.

$k = 5$.

Let K_{8m+5} be as in (2.7). The graph K_{8m+1} has an F_3 -decomposition. In the case $k = 5$ we saw that $K_{4,8m} \cup K_4$ has an F_3 -packing leaving two non-packed edges. Let $V(K_4) = \{a, b, c, d\}$ and $V(K_{8m+1}) = Z_{8m+1}$. Denote the non-packed edges by (a, d) and $(a, 8m-1)$. The F_3 graph in $K_{4,1} \cup \{(a, d), (a, 8m-1)\}$ is $(b, 8m, a; d, 8m-1)$. The edges $(d, 8m)$ and $(c, 8m)$ remain non-packed.

The proofs for $k = 6, 7$ are accomplished in the same ways as for F_2 .

Once again all cases have been considered and the proof is complete for F_3 .

F_4 : it is easy to see that the theorem does not hold for $n = 5$ and $n = 6$. For K_7 the graphs F_4 of the packing are: $(0; 1, 2, 3, 4)$, $(1; 2, 3, 4, 5)$, $(2; 3, 4, 5, 6)$, $(5; 4, 3, 0, 6)$, $(6; 0, 1, 3, 4)$. The edge $(3, 4)$ is left non-packed. Hence, the theorem is proved for $n = 7$. For $n = 8m, 8m+1$ we have an F_4 -decomposition [5, 6]. Hence, we again have to prove the theorem for the cases:

$$n = 8m + k, \quad k = 2, \dots, 7, \quad m \geq 1.$$

$k = 2$.

Let $K_{8m+2} = K_{8m+1} \cup K_{1,8m+1}$. The graph K_{8m+1} has an F_4 -decomposition. $K_{1,8m+1}$ is a star that can easily be packed by $2m$ stars F_4 , leaving one non-packed edge.

$k = 3$.

Let $K_{8m+3} = K_{8m} \cup K_{3,8m} \cup K_3$. The graph K_{8m} has an F_4 -decomposition. Let $K_{3,8m} = 3K_{1,8m}$. Since the graph $K_{1,8m}$ can be decomposed into $2m$ stars F_4 , it follows that $K_{3,8m}$ also has an F_4 -decomposition. Let $V(K_3) = \{a, b, c\}$, and create a decomposition of $K_{3,8m}$ which includes the three stars $(a; x, y, z, u)$, $(b; x, y, z, u)$, $(c; x, y, z, u)$. Replace the edge (a, u) by (a, b) , the edge (b, u) by (b, c) , and the edge (c, u) by (c, a) . We did not spoil any star of the decomposition of $K_{3,8m}$ and the star $(u; a, b, c)$ of three branches is left non-packed.

$k = 4$.

Let $K_{8m+4} = K_{8m} \cup K_{4,8m} \cup K_4$. The graph K_{8m} has an F_4 -decomposition. Let $K_{4,8m} = 4K_{1,8m}$. The graph $K_{1,8m}$ can be decomposed into $2m$ stars F_4 so $K_{4,8m}$ has an F_4 -decomposition. Let $V(K_4) = \{a, b, c, d\}$, and consider the subgraph $K_{4,4}$ of $K_{4,8m}$ whose vertices are given by $V(K_{4,4}) = \{a, b, c, d\} \cup \{8m-1, 8m-2, 8m-3, 8m-4\}$. The F_4 decomposition of $K_{4,8m}$ can be arranged in such a way that our $K_{4,4}$ is made up of the four F_4 graphs $(a; 8m-1, 8m-2, 8m-3, 8m-4)$, $(b; 8m-1, 8m-2, 8m-3, 8m-4)$, $(c; 8m-1, 8m-2, 8m-3, 8m-4)$ and $(d; 8m-1, 8m-2, 8m-3, 8m-4)$. Replace the edges $(a, 8m-1)$, $(b, 8m-1)$, $(c, 8m-1)$, $(d, 8m-1)$ with the edges (a, c) , (a, b) , (b, c) , (c, d) , respectively we now have a new F_4 graph, namely $(8m-1; a, b, c, d)$. The edges (a, d) and (b, d) are the only one which remain non-packed.

$k = 5$.

Let $K_{8m+5} = K_{8m} \cup K_{5,8m} \cup K_5$. As before K_{8m} and $K_{5,8m}$ have F_4 -decompositions. Now $K_5 = K_4 \cup K_{1,4}$ so we can complete the proof in the same way as in the case $k = 4$.

$k = 6$.

Let $K_{8m+6} = K_{8m} \cup K_{6,8m} \cup K_6$. The graphs K_{8m} and $K_{6,8m}$ have F_4 -decompositions. Let $V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Graph K_6 can be packed with the two F_4 $\{v_1, v_2, v_3, v_4, v_5\}$ and $\{v_2, v_3, v_4, v_5, v_6\}$. The induced graph on $\{v_3, v_4, v_5, v_6\}$ is K_4 . Hence, we can complete the proof here as in the case $k = 4$, leaving the edges (v_5, v_6) , (v_4, v_6) non-packed. Those edges together with the non-packed edge (v_1, v_6) accomplish the proof of the theorem in this case.

$k = 7$.

Let $K_{8m+7} = K_{8m} \cup K_{7,8m} \cup K_7$. The graphs K_{8m} and $K_{7,8m}$ have F_4 -decompositions and we apply the F_4 -packing shown for K_7 at the beginning of this case.

This completes the proof of the theorem for F_4 . $\square$

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