TAUBERIAN THEOREM FOR THE DISTRIBUTIONAL
STIELTJES TRANSFORMATION

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ABSTRACT. In this paper we use the notion of L-quasiasymptotic at infinity of dis-
tributions to obtain a final value Tauberian theorem for the distributional Stieltjes
transformation.

KEY WORDS AND PHRASES. Slowly varying function, the quasiasymptotic behaviour at in-
finity, distributional Stieltjes transform.

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1. NOTIONS AND NOTATION.

Throughout this paper, L will denote real valued, positive and measurable func-
tion on [0,∞) such that

$$\lim_{t \to \infty} \frac{L(t)}{e(t)} = 1$$

for each λ > 0. Function L which is regularly varying with index of regular varia-
tion α = 0, is called slowly varying and the class of such functions is introduced
and investigated by J. Karamata.

The quasiasymptotic behaviour at infinity of tempered distributions with supports
in [0,∞) (denoted by S_0^+) was defined by Zavijalov (see for instance [1]).

Definition 1. A distribution f ∈ S_0^+ has the L-quasiasymptotic at infinity of
the power a ∈ R and with the limit g ∈ S_0^+ , g ≠ 0, if for every φ ∈ S (S is the
space of rapidly decreasing functions)

$$\lim_{k \to \infty} \frac{\langle f(kt), \phi(t) \rangle}{k^a} = \Delta \langle g(t), \phi(t) \rangle.$$ 

From the properties of homogeneous distributions it follows ([1]) that if this limit
exists then g is a homogeneous distribution of degree a. Namely, g(t) = C f_{a+1}(t)
for some C ≠ 0, where

$$f_{a+1}(t) = \begin{cases} 
\frac{H(t) t^a}{\Gamma(a+1)} & : a > -1 \\
\partial^n f_{a+n+1}(t) & : a \leq -1, a+n > -1.
\end{cases}$$
As usual, \( H \) is the characteristic function of the interval \((0, \infty)\) and \( D \) stands for the distributional derivative.

We use the definition of the distributional Stieltjes transform given in [2], [3] in a little different notation ([4]).

Space \( J'(\rho), \rho \in \mathbb{R} \setminus -N_0 \) \((N_0 = \mathbb{N} \cup \{0\})\) is the space of distributions with supports in \([0, \infty)\) such that \( f \in J'(\rho) \) iff there exist \( k \in N_0 \) and a locally integrable function \( F \) with the support in \([0, \infty)\) such that
\[
(a) \ f = D^kF; \quad (b) \ \int_0^\infty |F(t)|^{(t+\beta)-\rho-k}dt < \infty \text{ for } \beta > 0 \quad (1.1)
\]
(D is the distributional derivative).

If instead of \((b)\) we suppose that there exist \( C = C(F) \) and \( \varepsilon = \varepsilon(F) > 0 \) such that
\[
(c) \ |F(x)| \leq C(1+x)^{\rho+k-1-\varepsilon} \text{ if } x > 0,
\]
the corresponding space is denoted by \( I'(\rho) \).

It was proved in [4] that:

If \( f \in J'(\rho), \rho+k > 0 \) \((k \text{ is from (a)})\), then \( f \in I'(\beta) \) for \( \beta > \rho \) and \( \beta \in \mathbb{R} \setminus -N_0 \).

(1.2)

If \( f \in J'(\rho), \rho+k > 0 \), then \( f \in I'(\beta) \) for \( \beta > -k \) \((k \text{ is from (a)})\) and \( \beta \in \mathbb{R} \setminus -N_0 \).

The Stieltjes transform \( S_\rho \) of index \( \rho \), \( \rho \in \mathbb{R} \setminus -N_0 \) of a distribution \( f \in J'(\rho) \) with the properties given in (1.1) is a complex-valued function given by
\[
(S_\rho(f))(s) = (\rho)_k \int_0^\infty \frac{F(t)dt}{(t+s)^{\rho+k}}, \text{ s } \in \mathbb{C} \setminus (-\infty,0] \quad (1.3)
\]

where \( (\rho) = \rho(\rho+1) \ldots (\rho+k-1), k > 0 \) and \( (\rho)_0 = 1 \).

It is proved in [3] that \((S_\rho(f))(s)\) is a holomorphic function of the complex variable \( s \) in the domain \( \mathbb{C} \setminus (-\infty,0) \) provided that \( f \in J'(\rho) \).

Observe that \( f \in J'(\rho) \) implies that \( f \in J'(\rho+n), n \in \mathbb{N} \).

The following equality holds:
\[
(\rho)_n(S_\rho^n(f))(s) = (S_\rho(D^n f))(s), f \in J'(\rho) \text{ and } n \in \mathbb{N}.
\]

We shall need the following theorem ([5], p. 339, Macaev and Palant)

THEOREM A. Let us suppose that for some \( m > 0 \) and \( x \in \mathbb{R} \)
\[
\int_0^\infty \frac{d\phi(\lambda)}{(\lambda+x)^{m+1}} \quad \int_0^\infty \frac{d\psi(\lambda)}{(\lambda+x)^{m+1}}
\]
and the following conditions are satisfied:

1) Functions \( \phi \) and \( \psi \) are defined for \( x > 0 \) and are non-decreasing;

2) \( \lim_{x \to \infty} x = \infty \);

3) For any \( C > 1 \) there are constants \( \gamma \) and \( N \), \( 0 < \gamma < m, N > 0 \), such that for any \( x < y < N \) is
\[
\frac{\phi(x)}{\phi(y)} \leq C \left( \frac{x}{y} \right)^{\gamma}.
\]
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Then, for \( \lambda \to \infty \), \( \phi(\lambda) \sim \psi(\lambda) \). (This means \( \left| \frac{\phi(\lambda)}{\psi(\lambda)} - 1 \right| < \varepsilon \) if \( \lambda = \lambda_0(\varepsilon) \).)

2. TAUBERIAN THEOREM.

THEOREM. Let us suppose that \( s > 1 \), \( \rho + k - s - 1 > 0 \), \( f \in I'(\rho) \) and \( F \) (from (1.1)(a)) is non-decreasing function. Moreover, let

\[
(S_{\rho}(f))(x) = \frac{1}{x^s L(x)}, \quad x^{\infty}
\]

where \( L \) is slowly varying function in some interval \([A, \infty)\), such that \( x^{\rho + k - s - 1} L(x) \) is non-decreasing function.

Then \( f \) has the \( L \)-quasiasymptotic of the power \( \rho - s \) and with the limit

\[
\left( \frac{\rho - s}{\rho} \right)_k \frac{1}{x^{\rho - s}}.
\]

PROOF. Let us put

\[
\phi(\lambda) = \begin{cases} \frac{1}{x^{\rho + k - s - 1} L(x)} & ; \quad x > A \\ 0 & ; \quad x \leq A \end{cases}
\]

Then \( \phi \) has the \( L \)-quasiasymptotic of the power \( \rho + k - s - 1 \) (Theorem 1) and with the limit \( x^{\rho + k - s - 1} \). By (6) it is

\[
\int_0^\infty \frac{d\phi(t)}{(x+t)^{\rho + k - 1}} = \left( \frac{\rho - s}{\rho} \right)_k \int_0^\infty \frac{d\phi(t)}{(x+t)^{\rho + k - 1}} \sim \frac{1}{x^s L(x)}, \quad x^{\infty}.
\]

Now we show that the conditions of Theorem A hold for \( \phi \) and \( F \). In fact we have only to show that for some \( 0 < \gamma < \rho + k - 2 \) and every \( C > 1 \) there exists \( N > 0 \) such that

\[
\frac{\phi(\lambda y)}{\phi(y)} < C \lambda^\gamma \quad \text{for} \quad \lambda > 1 \quad \text{and} \quad y > N.
\]

Let us put \( \gamma = \rho + k - s - 1 + \varepsilon \) where we choose \( \varepsilon > 0 \) such that \( \gamma > 0 \) and \( \varepsilon < s - 1 \). After substituting (2.1) in (2.3) we obtain

\[
L(\lambda y) \leq C \lambda^\varepsilon L(y)
\]

and this inequality is true if \( \lambda > 1 \) and \( y > N \) where \( N \) depends on \( C \) (see [6]).

From the assumption that \( f \in I'(\rho) \) and (2.2) we have

\[
(S_{\rho}(f))(x) = \left( \frac{\rho - s}{\rho} \right)_k \int_0^\infty \frac{dF(t)}{(x+t)^{\rho + k - 1}} = \left( \frac{\rho - s}{\rho} \right)_k \int_0^\infty \frac{dF(t)}{(x+t)^{\rho + k - 1}} \sim \frac{1}{x^s L(x)}, \quad x^{\infty}.
\]

It implies

\[
\left( \frac{\rho - s}{\rho} \right)_k \int_0^\infty \frac{dF(t)}{(x+t)^{\rho + k - 1}} \sim \int_0^\infty \frac{d\phi(t)}{(x+t)^{\rho + k - 1}}, \quad x^{\infty}
\]

and by Theorem A it implies

\[
\left( \frac{\rho - s}{\rho} \right)_k \sim \phi, \quad x^{\infty}.
\]

Thus, we obtain that \( F(x) \sim \frac{1}{x^s L(x)}, \quad x^{\infty} \).

Since \( \rho + k - s - 1 > 0 \), it follows ([1]) that \( F \) has the \( L \)-quasiasymptotic of the power \( \rho + k - s - 1 \) and with the limit \( \frac{1}{x^s L(x)} \). Since \( f = D^k F \) it easily follows (1) that \( f \) has the \( L \)-quasiasymptotic of the power \( \rho - s - 1 \) and with the limit

\[
\left( \frac{\rho - s}{\rho} \right)_k \frac{1}{x^{\rho - s - 1}}.
\]

\[
\Delta
\]
COROLLARY. Let us suppose that \( f \in \mathcal{J}'(\rho) \) and for some \( \rho \geq \rho \) \( \rho \in \mathbb{R}\setminus\{-N_0\} \)

\[
(S_\rho f)(x) \sim \frac{1}{x^{s} L(x)}, \quad x \to \infty, \quad s > 1,
\]

where \( L \) is a slowly varying function on \([A, \infty)\). Further, suppose \( \tilde{\rho} - s + k > 0 \) and \( x^{\tilde{\rho} - s + k} L(x) \) is non-decreasing in \([A, \infty)\). (Consequently, \( f \in I'(\tilde{\rho}) \) and \( f(x) = D^{k+1} \left( \int_0^x F(t) dt \right) \)).

Then \( f \) has the \( L \)-quasiasymptotic of the power \( \tilde{\rho} - 1 - s \) and with the limit \( \lim_{x \to \infty} \frac{x^{\tilde{\rho} - s}}{x^{\tilde{\rho} + k+1}} \)

REFERENCES
