P-REPRESENTABLE OPERATORS IN BANACH SPACES

ROSHDI KHALIL

Department of Mathematics
The University of Michigan
Ann Arbor, Michigan, 48109, U.S.A.

(Received November 7, 1985)

ABSTRACT. Let E and F be Banach spaces. An operator $T \in L(E, F)$ is called p -representable if there exists a finite measure μ on the unit ball, $B(E^*)$, of E^* and a function $g \in L^q(\mu, F)$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$Tx = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$$

for all $x \in E$. The object of this paper is to investigate the class of all p -representable operators. In particular, it is shown that p -representable operators form a Banach ideal which is stable under injective tensor product. A characterization via factorization through L^p -spaces is given.

KEY WORDS AND PHRASES. Representable Operator, Banach Space, Stable Ideal Operators.
1980 AMS SUBJECT CLASSIFICATION CODE. 47B10.

1. INTRODUCTION.

Let $L(E, F)$ be the space of all bounded linear operators from E into F , and $B(E^*)$ the unit ball of E^* , the dual of E . The completion of the injective tensor product of E and F is denoted by $E \hat{\otimes} F$. Integral operators in $L(E, F)$ were first defined by Grothendieck, [2], as those operators which can be identified with elements in $(E \hat{\otimes} F)^*$. These operators turn to have a nice integral representation. We refer to Jarchow, [4], for statements and proofs of such representations. Later on, Persson and Pietsch, [5], defined p -integral operators in $L(E, F)$ as those operators $T: E \rightarrow F$ such that $Tx = \int_{B(E^*)} \langle x, x^* \rangle dG(x^*)$, for all $x \in E$ where G is a vector measure on $B(E^*)$ with values in F and

$$\left\| \int_{B(E^*)} \varphi(x^*) dG(x^*) \right\| \leq \left(\int_{B(E^*)} |\varphi(x^*)|^p d\mu \right)^{1/p} \text{ for some finite measure } \mu \text{ on } B(E^*)$$

and all continuous functions φ on $B(E^*)$. The representing vector measure for T need not be of bounded variation. Further, if G is of bounded variation and F doesn't have the Radon-Nikodym property, then T need not be a kernel integral operator.

The object of this paper is to study operators which are in some sense kernel integral operators. Such operators is a sub-class of Pietsch p -integral operators.

Throughout this paper, if E is a Banach space, then E^* is the dual of E and $B(E)$ the closed unit ball of E . If K is a set then 1_K is the characteristic function of K . If (Ω, μ) is a measure space, then $L^p(\Omega, \mu, E)$ is the space of

all p -Bochner integrable functions defined on Ω with values in E , for $1 \leq p < \infty$. If $p = \infty$, $L^\infty(\Omega, \Sigma, \mu, E)$ is the space of Σ -essentially bounded functions on Ω with values in E . The real $1 \leq q \leq \infty$ always denote the conjugate of p : $\frac{1}{p} + \frac{1}{q} = 1$. Most of our terminology and notations are from Pietsch [6] and Diestel and Uhl [1]. We refer to these texts for any notion cited but not defined in this paper.

2. $R_p(E, F)$.

DEFINITION 2.1. An operator $T \in L(E, F)$ is called p -representable operator if there exists a finite measure μ defined on the Borel sets of $B(E^*)$ and a function $g: B(E^*) \rightarrow F$ such that $\int_{B(E^*)} \|g(x^*)\|^q d\mu(x^*) < \infty$, and $Tx = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$ for all $x \in E$.

It follows from the definition that every p -representable operator is Pietsch- p -integral operator, but not the converse. Let $R_p(E, F)$ be the set of all p -representable operators from E into F .

LEMMA 2.2. $R_p(E, F)$ is a vector space.

PROOF. Let $T_1, T_2 \in R_p(E, F)$ such that

$$T_i(x) = \int_{B(E^*)} \langle x, x^* \rangle g_i(x^*) d\mu_i(x^*).$$

Set $\mu = \mu_1 + \mu_2$. Then $\mu_i \ll \mu$. Consequently, $d\mu_i = f_i d\mu$. Further, since $\mu_i(K) \leq \mu(K)$ for all Borel sets K on $B(E^*)$, it follows that $0 \leq f_i(x^*) \leq 1$ a.e. μ , $i = 1, 2$. Let $\tilde{g}(x^*) = g_1(x^*)f_1(x^*) + g_2(x^*)f_2(x^*)$. Since $1 \leq p < \infty$, and $0 \leq f_i(x^*) \leq 1$, we have $\tilde{g} \in L^q(B(E^*), \mu, F)$. Further $(T_1 + T_2)(x) = \int_{B(E^*)} \langle x, x^* \rangle \tilde{g}(x^*) d\mu$, for all $x \in E$. This ends the proof.

For $T \in R_p(E, F)$, we define

$$\|T\|_{\sigma(p)} = \inf \left\{ \left(\int \|g(x^*)\|^q d\mu(x^*) \right)^{1/q} \right\}$$

where the infimum is taken over all g and μ for which $T(x) = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$, $x \in E$. It is not difficult to show that $\|\cdot\|_{\sigma(p)}$ is a norm on $R_p(E, F)$.

LEMMA 2.3. For $T \in R_p(E, F)$, $\|T\| \leq \|T\|_{\sigma(p)}$.

PROOF. Let $Tx = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$ for some μ and g as in the Definition 2.1. Choose g and μ such that $\left(\int \|g(x^*)\|^q d\mu(x^*) \right)^{1/q} \leq \|T\|_{\sigma(p)} + \epsilon$, for a given small $\epsilon > 0$. Then, using Holder's inequality:

$$\begin{aligned} \|Tx\| &\leq \left(\int_{B(E^*)} \|g(x^*)\|^q d\mu(x^*) \right)^{1/q} \\ &\leq \|T\|_{\sigma(p)} + \epsilon. \end{aligned}$$

Hence $\|T\| \leq \|T\|_{\sigma(p)} + \epsilon$. Since ϵ is arbitrary, the result follows.

LEMMA 2.4. Every element $T \in R_p(E, F)$ is an approximable operator in $L(E, F)$.

PROOF. Let $Tx = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$, for some finite measure on $B(E^*)$ and some $g \in L^q(B(E^*), \mu, F)$. Choose ν and g such that

$$\left(\int_{B(E^*)} \|g(x^*)\|^q d\mu(x^*) \right)^{1/q} \leq \|T\|_{\sigma(p)} + \varepsilon.$$

Let g_n be a sequence of simple functions in $L^q(B(E^*), \mu, F)$ such that

$\int_{B(E^*)} \|g(x^*) - g_n(x^*)\|^q d\mu(x^*) \rightarrow 0$. Define $T_n(x^*) = \int_{B(E^*)} \langle x, x^* \rangle g_n(x^*) d\mu(x^*)$. Then each T_n is a finite rank operator, and $\|T - T_n\|_{\sigma(p)} \rightarrow 0$. Then by definition of approximable operators, Pietsch [6], T is approximable. This ends the proof.

THEOREM 2.5. Let H, E, F and G be Banach spaces, and $T \in R_p(E, F)$, $A \in L(F, G)$ and $B \in L(H, E)$. Then $ATB \in R_p(H, G)$ and $\|ATB\|_{\sigma(p)} \leq \|A\| \|B\| \|T\|_{\sigma(p)}$.

PROOF. Let $Tx = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$ for all $x \in E$ and some finite measure μ on $B(E^*)$ and some $g \in L^q(B(E^*), \mu, F)$. Then

$$ATx = \int_{B(E^*)} \langle x, x^* \rangle Ag(x^*) d\mu(x^*)$$

and $\int_{B(E^*)} \|Ag(x^*)\|^q d\mu(x^*) \leq \|A\|^q \int_{B(E^*)} \|g(x^*)\|^q d\mu(x^*)$. Hence $AT \in R_p(E, G)$ and $\|AT\|_{\sigma(p)} \leq \|A\| \|T\|_{\sigma(p)}$.

To show $TB \in R_p(H, F)$, let g_n be a sequence of simple functions converging to g in $L^q(B(E^*), \mu, F)$, and T_n be the associated operators in $R_p(E, F)$. So

$$T_n x = \int_{B(E^*)} \langle x, x^* \rangle g_n(x^*) d\mu(x^*).$$

With no loss of generality we assume $\|B\| = 1$. Define the vector measures G on $B(H^*)$ into F via:

$$\begin{aligned} G_n(K) &= \int_{B(E^*)} 1_K(y^*) dG(y^*) \\ &= \int_{B(E^*)} 1_K(B^*x^*) S_n(x^*) d\mu(x^*). \end{aligned}$$

Clearly, G_n is a countably additive vector measure of bounded variation. Further, if we define the measure ν on $B(H^*)$ via

$$\nu(K) = \int_{B(E^*)} 1_K(B^*x^*) d\mu(x^*),$$

then, using Holder's inequality:

$$\|G_n(K)\| \leq \left(\int_{B(E^*)} \|g_n(x^*)\|^q d\mu(x^*) \right)^{1/q} \cdot [\nu(K)]^{1/p}.$$

Hence $G_n \ll \nu$. Since the range of G_n is finite dimensional, it has the Radon-Nikodym property, and consequently there exists $S_n \in L^1(B(H^*), \nu, F)$ such that $dG_n = S_n d\nu$. Further, it is easy to check that $S_n \in L^q(B(H^*), \nu, F)$.

An application of the Hahn-Banach theorem, we get:

$$\begin{aligned} T_n B y &= \int_{B(E^*)} B y, x^* \cdot g_n(x^*) d\mu(x^*) \\ &= \int_{B(H^*)} \langle y, y^* \rangle S_n(y^*) d\nu(y^*). \end{aligned}$$

Since the function $B y, x^*$ is bounded on $B(E^*)$, the sequence $(\langle B y, \cdot \rangle g_n)$ is Cauchy in $L^q(B(E^*), \mu, F)$. Consequently the sequence $(\langle y, \cdot \rangle S_n)$ is Cauchy in $L^q(B(H^*), \nu, F)$. Let $\langle y, \cdot \rangle S$ be the limit of $(\langle y, \cdot \rangle S_n)$ in $L^q(B(H^*), \nu, F)$. It is not difficult to see that $T_n B$ converges in the operator norm to the operator $J y = \int_{B(H^*)} \langle y, y^* \rangle S(y^*) d\nu(y^*)$.

However $T_n B \rightarrow TB$ in the operator norm. Hence $TBy = \int_{B(H^*)} \langle y, y^* \rangle S(y^*) d\mu(y^*)$, and $TB \in R_p(H, F)$. Further $\|TB\| \leq \|T\|_{\sigma(p)} \|B\|$. This ends the proof.

Theorem 2.5 states that $(R_p, \|\cdot\|_{\sigma(p)})$ is a normed operator ideal, [6].

DEFINITION 2.6. Let (Ω, μ) be a measure space and F a Banach space. An operator $T \in L(L^p(\Omega, \mu), F)$ is called B-vector integral operator if there exists $g \in L^q(\Omega, \mu, F)$ such that

$$Tf = \int_{\Omega} f(t)g(t)d\mu(t)$$

for all $f \in L^p(\Omega, \mu)$.

If the function g is only Pettis q -integrable and the integral defining Tf is the Pettis integral, then T is known to be called vector integral operator [1].

Now using Theorem 2.5 we can prove:

THEOREM 2.7. Let E, F be Banach spaces and $T \in L(E, F)$. The following are equivalent:

(i) $T \in R_p(E, F)$

(ii) There exists operators $T_1 \in L(E, L^p(\Omega, \mu))$ and $T_2 \in L(L^p(\Omega, \mu), F)$ for some measure space (Ω, μ) such that T_2 is B-vector integral operator and $T = T_2 T_1$.

PROOF. (i) $\rightarrow$ (ii). Let $T \in R_p(E, F)$ and

$$Tx = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$$

for some finite measure μ on $B(E^*)$ and $g \in L^q(B(E^*), \mu, F)$. Define

$$T_1: E \longrightarrow L^p(B(E^*))$$

$$(T_1 x)(x^*) = \langle x, x^* \rangle,$$

and

$$T_2: L^p(B(E^*), \mu) \longrightarrow F$$

$$T_2(f) = \int_{B(E^*)} f(x^*)g(x^*)d\mu.$$

Then T_2 is a B-vector integral operator and $T = T_2 T_1$.

(ii) $\rightarrow$ (i). Let $T = T_2 T_1$, $T_1 \in L(E, L^p(\Omega, \mu))$ and T_2 is a B-vector integral operator in $L(L^p(\Omega, \mu), F)$. Then $T_2 \in R_p(L^p(\Omega, \mu), F)$. Using Theorem 2.6, $T_2 T_1 \in R_p(E, F)$. This ends the proof.

Let $I_p(E, F)$ be the space of Pietsch p -integral operators from E into F , and $\|T\|_{i(p)}$ be the p -integral norm for $T \in I_p(E, F)$. Clearly $R_p(E, F) \subseteq I_p(E, F)$ and $\|T\|_{i(p)} \leq \|T\|_{\sigma(p)}$ for all $T \in R_p(E, F)$. This, together with the fact that $I_p(E, F)$ is complete, [5], one can prove:

THEOREM 2.8. $(R_p(E, F), \|\cdot\|_{\sigma(p)})$ is a Banach space.

If F has the Radon Nikodym property, then $R_1(E, F) = I_1(E, F)$, and by using Corollary 5 in [1], we see that $R_1(C(\Omega), F) = I_1(E, F) = N_1(C(\Omega), F)$, where $N_1(E, F)$ is the class of nuclear operators from E into F .

Further if $\pi_p(E, F)$ is the class of p -summing operators from E into F , then it follows from the Grothendieck-Pietsch representation theorem [6], that $R_p(E, F) \subseteq \pi_p(E, F)$

3. IDEAL PROPERTIES OF R_p .

We let R_p denote the operator ideal of all p -representable operators. The following notions are taken from Pietsch [5] and Holub [3].

(i) An operator ideal J is called regular if for all Banach spaces E and F , $T \in J(E, F)$ if and only if $K_F T \in J(E, F^{**})$, where K_F is the natural embedding of F into F^{**} .

(ii) J is called closed if the closure of $J(E, F)$ in $L(E, F)$ is $J(E, F)$ for all Banach spaces E and F .

(iii) J is called injective if whenever $J_F T \in J(E, \ell^\infty(B(F^*)))$, then $T \in J(E, F)$ for all Banach spaces E and F . Here J_F is the natural embedding of F into $\ell^\infty(B(F^*))$.

(iv) J is called stable with respect to the injective tensor product if $T_i \in J(E_i, F_i)$, then $T_1 \otimes T_2 \in J(E_1 \otimes E_2, F_1 \otimes F_2)$, for all Banach spaces E_1, E_2, F_1, F_2 .

THEOREM 3.1. R_p is regular.

PROOF. Let E and F be any Banach spaces and let $K_F T \in R_p(E, F^*)$, for $T \in L(E, F)$. Then $K_F T x = \int_{B(E^*)} \langle x, x^* \rangle g(x^*) d\mu(x^*)$ for some μ and g as in Definition 2.1.

Now $g(x^*) \in K_F(F)$ for all $x^* \in B(E^*)$. Since $K_F: F \rightarrow K_F(F)$ is an isometric onto operator, the function $g(x^*) = K_F^{-1}(g(x^*))$ is well defined measurable and $\tilde{g} \in L^Q(B(E^*), \mu, F)$. Further

$$Tx = \int_{B(E^*)} \langle x, x^* \rangle \tilde{g}(x^*) d\mu(x^*).$$

Hence $T \in R_p(E, F)$. This ends the proof.

In a similar way one can prove:

THEOREM 3.2. R_p is injective

THEOREM 3.3. R_p is stable.

PROOF. Let $T_i \in R_p(E_i, F_i)$, $i = 1, 2$ and

$$\begin{aligned} T_1 x &= \int_{B(E_1^*)} \langle x, x^* \rangle g_1(x^*) d\mu_1(x^*) \\ T_2 x &= \int_{B(E_2^*)} \langle x, x^* \rangle g_2(x^*) d\mu_2(x^*), \end{aligned}$$

where μ_i and g_i be the associated measures and functions as in Definition 2.1. If $E_i \otimes F_i$, $i = 1, 2$, is the completion of the injective tensor product of E_i with F_i , [1], then $T_1 \otimes T_2 \in L(E_1 \otimes F_1, E_2 \otimes F_2)$. Further:

$$(T_1 \otimes T_2)(x \otimes y) = \left(\int_{B(E_1^*)} \langle x, x^* \rangle g_1(x^*) d\mu_1(x^*) \right) \left(\int_{B(E_2^*)} \langle y, y^* \rangle g_2(y^*) d\mu_2(y^*) \right).$$

Let K be the w^* -closure of $B(E_1^*) \otimes B(E_2^*) = \{x^* \otimes y^*: x^* \in B(E_1^*), y^* \in B(E_2^*)\}$ in $(E_1 \otimes E_2)^*$. Since the map $\gamma: E_1^* \otimes E_2^* \rightarrow E_1^* \otimes E_2^*$, the projective tensor product of E_1 with E_2 , is continuous, [7], it follows that the map $\gamma: B(E_1^*) \times B(E_2^*) \rightarrow B(E_1^*) \otimes B(E_2^*)$, $\gamma(x^*, y^*) = x^* \otimes y^*$ is continuous. This induces an isometric into

operator $\psi: C(K) \longrightarrow C(B(E_1^*) \vee B(E_2^*))$ defined by $\psi(f) = f \circ \gamma$. Consequently, there exists a measure μ on K such that

$$\int_K f(z^*) d\mu(z^*) = \int_{B(E_1^*) \vee B(E_2^*)} f \circ \gamma(x^*, y^*) d(\mu_1 \times \mu_2)(x^*, y^*).$$

Extend μ to $B(E_1 \oplus E_2)^*$ by putting $\mu \equiv 0$ on $B(E_1 \oplus E_2)^* \setminus K$. Further define $g: B(E_1 \oplus E_2)^* \longrightarrow F_1 \oplus F_2$ via $g(x^* \oplus y^*) = g_1(x^*) \oplus g_2(y^*)$ if $x^* \in B(E_1^*)$, $y^* \in B(E_2^*)$, and $g(z^*) = 0$ otherwise. Then it is not difficult to see that

$$(T_1 \oplus T_2)(z) = \int_{B(E_1 \oplus E_2)^*} \langle z, z^* \rangle g(z^*) d\mu(z^*)$$

for all $z \in E_1 \oplus E_2$. Since $g \in L^q(B(E_1 \oplus E_2)^*, \mu, F_1 \oplus F_2)$, it follows that $T_1 \oplus T_2 \in R_p(E_1 \oplus E_2, F_1 \oplus F_2)$. This ends the proof.

A negative result for R_p is the following:

THEOREM 3.4. R_p is not closed.

PROOF: Assume R_p is closed. Since the ideal of finite rank operator is contained in R_p , one has the ideal of approximable operators is contained in R_p . By Lemma 2.4, one gets $R_p =$ the ideal of approximable operators. Theorem 2.8, together with the open mapping theorem we get that $\|\cdot\|_{\sigma(p)}$ and $\|\cdot\|$ are equivalent on R_p . This is a contradiction. Hence R_p is not closed.

ACKNOWLEDGEMENT. The author would like to thank Professor Ramanujan and Dr. Defant for stimulating discussions. This work was done while the author was a visiting professor at the University of Michigan. The author would also like to thank the Department of Mathematics at the University of Michigan for their warm hospitality.

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