A GENERALIZATION OF A THEOREM
BY CHEO AND YIEN CONCERNING DIGITAL SUMS

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ABSTRACT. For a non-negative integer n, let s(n) denote the digital sum of n. Cheo
and Yien proved that for a positive integer x, the sum of the terms of the sequence
\{s(n) : n = 0, 1, 2, ..., (x-1)\}
is \((4.5)x \log x + O(x)\). In this paper we let k be a positive integer and determine that
the sum of the sequence
\{s(kn) : n = 0, 1, 2, ..., (x-1)\}
is also \((4.5)x \log x + O(x)\). The constant implicit in the big-oh notation is dependent
on k.

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1. INTRODUCTION.

In Cheo and Yien [1], it was proven that for a positive integer x,
\[ \sum_{n = 0}^{x-1} s(n) = (4.5)x \log x + O(x) \] (1.1)
where \(s(n)\) denotes the digital sum of n. Here, we will show that, in fact, for any
positive integer k,
\[ \sum_{n = 0}^{x-1} s(kn) = (4.5)x \log x + O(x) \] (1.2)
where the constant implicit in the big-oh notation is dependent on k.

The following notation will be used to facilitate the proof of (1.2). For integers
x and y,
\[ x \mod y \] (1.3)
will be the remainder when x is divided by y and, as usual, square brackets will denote
the integral part operator. In addition, for non-negative integers m, i, and j we let
\[ [m]_j = m \mod 10^i, \] (1.4)
\[ [m]_i = [m/10^i], \] (1.5)
and

\[ [m]^j_i = \left[ \left[ m \right]^j \right]_i \]  \tag{1.6} \]

for \( i < j \).

Thus, the j right-most digits of m are given by (1.4) and the number determined by dropping the i right-most digits of m is given by (1.5). Therefore, the number determined from the jth right-most digit of m to the \((i + 1)\)st right-most digit of m is given by (1.6).

2. A PROOF OF (1.2) WHEN \( k \) AND 10 ARE RELATIVE PRIME.

Let \( (k, 10) = 1 \), \( x \) be a positive integer, and \( L = \lfloor \log x \rfloor \). Then

\[ \sum_{n=0}^{x-1} s(kn) = \sum_{n=0}^{x-1} s(\lfloor kn \rfloor) + \sum_{n=0}^{x-1} s(\lfloor kn \rfloor, L) \]  \tag{2.1} \\
\[ = \sum_{n=0}^{x-1} s(\lfloor kn \rfloor, L) + O(x) \]  \tag{2.2} \\

This follows since for non-negative integers \( L \) and \( m \),

\[ m = [m]^L + 10^L[m]_L \]  \tag{2.3} \\

and so

\[ s(m) = s([m]^L) + s([m]_L) \]  \tag{2.4} \\

Also, since each \( s(\lfloor kn \rfloor, L) \) is bounded by a constant (dependent on \( k \)), we have that the second term of (2.1) is \( O(x) \).

Next, for \( i = 0, 1, 2, \ldots, L \) define

\[ x_i = [x]_{L+1-i} 10^{L+1-i} \]  \tag{2.5} \\

Then,

\[ \sum_{n=0}^{x-1} s(\lfloor kn \rfloor, L) = \frac{x-1}{x} \sum_{n=0}^{x-1} s(\lfloor kn \rfloor, L) + \sum_{n=0}^{x-1} s(\lfloor kn \rfloor, L) \]  \tag{2.6} \\
\[ = \frac{x-1}{x} s(\lfloor kn \rfloor, L) + \frac{x-1}{x} s(\lfloor kn \rfloor, L) + \frac{x-1}{x} s(\lfloor kn \rfloor, L). \]

In the same way,

\[ \sum_{n=x_1}^{x-1} s(\lfloor kn \rfloor, L-1) = \frac{x-1}{x} s(\lfloor kn \rfloor, L-1) + \frac{x-1}{x} s(\lfloor kn \rfloor, L-2) \]  \tag{2.7} \\
\[ + \frac{x-1}{x} s(\lfloor kn \rfloor, L-2). \]

Continuing in this manner and combining terms, we have

\[ \sum_{n=0}^{x-1} s(\lfloor kn \rfloor, L) = \sum_{i=1}^{L} \frac{x_i-1}{x_i} s(\lfloor kn \rfloor, L+1-i) \]  \tag{2.8} \\
\[ + \sum_{i=1}^{L} \frac{x_i-1}{x_i} s(\lfloor kn \rfloor, L-i). \]
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Since
\[ s([kn]_{L+1-i}) \]  \hspace{1cm} (2.9)

is a decimal digit and
\[ x - x_i = [x]_{L+1-i} \leq 10^{L+1-i} \]  \hspace{1cm} (2.10)

for each \( i \), it follows that
\[
\frac{L}{i} \sum_{n=x_i}^{x_i-1} s([kn]_{L+1-i}) = O(x) . \hspace{1cm} (2.11)
\]

To determine the value of the first term of (2.8), we need the following lemma. Its proof is straightforward and will not be given.

**LEMMA 2.** Let \( d \) and \( i \) be non-negative integers. Then for \( (k,10) = 1 \),
\[
\{[kn]_i : n = d,d+1, \ldots, d+10^i-1\} = \{n : n = 0,1, \ldots, 10^i-1\} . \hspace{1cm} (2.12)
\]

By this lemma and the fact that
\[
x - x_i - 10^{L+1-i} \hspace{1cm} (2.13)
\]

it follows that
\[
\frac{L}{i} \sum_{n=x_i-1}^{x_i-2} s([kn]_{L+1-i}) = \sum_{n=0}^{10^{L+1-i}} s(n) \hspace{1cm} (2.14)
\]

for each \( i \).

Now since
\[
10^{L+1-i} - 1 \sum_{n=0}^{10^{L+1-i}} s(n) = 4.5(L + 1 - i)10^{L+1-i} \hspace{1cm} (2.15)
\]

by [2], we have that
\[
\frac{L}{i} \sum_{n=x_i-1}^{x_i-2} s([kn]_{L+1-i}) = (4.5)x\log x + O(x) . \hspace{1cm} (2.16)
\]

Using (2.16) and (2.11) in (2.8), by (2.2) we have the expression given in (1.2). The constant implicit in the big-oh notation is dependent on \( k \) with \( k \) and \( 10 \) relatively prime.

3. CONCLUSION.

For any positive integer \( k \), there exists non-negative integers \( a, b, \) and \( r \) such that \( k = 2^a5^b r \) with \( (r,10) = 1 \). Note that if \( k = r \), then we have (1.2). However, by use of the following generalization to Lemma 2, and some technical modifications, it can be shown that the restriction that \( k \) and \( 10 \) be relatively prime can be removed in the derivation of (2.1). That is,
\[
x - 1 \sum_{n=0}^{x-1} s(kn) = (4.5)x\log x + O(x) \hspace{1cm} (3.1)
\]

for any positive integer \( k \).

**LEMMA 3.** Let \( k = 2^a5^b r \) with \( (r,10) = 1 \) and \( i \geq \max \{a,b\} \). Then for any non-
negative integer $d$,

$$\{[kn]^i : n = d, d+1, d+2, \ldots, d + (10^i/2a^b) - 1\}$$

$$= \{2a^bn : n = 0, 1, 2, \ldots, (10^i/2a^b) - 1\}.$$  \hspace{1cm} (3.2)

Finally, based on the above techniques, it is strongly conjectured that for any positive integers $k_1$ and $k_2$, it again follows that

$$\frac{x-1}{\log x} \sum_{n=0}^{\infty} s(k_1n + k_2) = (4.5)x\log x + O(x).$$  \hspace{1cm} (3.3)

REFERENCES


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