A NOTE ON THE INVERSE FUNCTION THEOREM
OF NASH AND MOSER

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ABSTRACT. The Nash-Moser inverse function theorem is proved for different kind of
differentiabilities.

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smoothing operators, tame map, $C^\infty_{\alpha}$-differentiability.

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1. INTRODUCTION

The purpose of this note is to formulate the inverse function theorem of Nash and
Moser for different differentiabilities using a categorical approach. The proof is
based on the inverse function theorem of Nash and Moser in the version of Hamilton [1]
formulated in the category of graded Fréchet spaces which admit smoothing operators and
$C^\infty_{\alpha}$-differentiable [2] tame maps. Our proof is using the same technique as Schmid [3]
uses when he proves this theorem for a stronger notion of differentiability, called
the $\Gamma$-differentiability, than the $C^\infty_{\alpha}$-differentiability. From our formulation it is
possible to derive the inverse function theorem of Nash and Moser for natural
differentiabilities stronger than the $C^\infty_{\alpha}$-differentiability.

2. THE INVERSE FUNCTION THEOREM OF NASH AND MOSER.

Let $\mathcal{L}C$ denote the category of locally convex limit vector spaces [2] and con-
tinuous linear mappings. Further let $K_{\alpha}$ denote a coreflective subcategory of $\mathcal{L}C$
which is closed under finite products and the coreflector $\gamma_{\alpha} : \mathcal{L}C \to K_{\alpha}$ is the identity
on morphisms and such that the identity mapping $(C^\alpha_{\alpha}(X,F))^\alpha = C^\alpha_{\alpha}(X,F) \to C^\alpha_{\alpha}(X,F)$ is
continuous. Here $C^\alpha_{\alpha}(X,F)$ denotes the vector space of continuous mappings $X \to F$
endowed with continuous convergence [2], and $X$ is a limit space and $F \in obj(\mathcal{L}C)$.

For any pair $E,F \in obj(\mathcal{L}C)$ we let $L^k_{\alpha}(E,F)$ be the space of all continuous
$k$-linear mappings from $E^k$ into $F$, endowed with continuous convergence. We write
$(L^k_{\alpha}(E,F))^\alpha = L^k_{\alpha}(E,F)$.

DEFINITION. Let $E$ and $F$ be locally convex spaces and let $U$ be open in $E$.  


A mapping \( f : U \to F \) is said to be **differentiable of class** \( C^p_\alpha \), if there exist functions
\[
D^k f : U \to L^k(E,F), \quad k = 0, 1, \ldots, p,
\]
such that \( D^0 f = f \) and for each \( x \in U \), each \( h \in E \) and each \( k = 0, 1, \ldots, p-1 \), we have
\[
\lim_{t \to 0} t^{-1}(D^k f(x + th) - D^k f(x)) = D^{k+1} f(x) h,
\]
and such that for each \( k \in \mathbb{N} \), \( k \leq p \), the following two conditions are satisfied:
\[
(1) \quad D^k f(U) \subseteq L^k(E,F) \quad \text{and} \quad (2) \quad D^k f : U \to L^k(E,F) \text{ is continuous.}
\]

\( f \) is called **differentiable of class** \( C^p_\alpha \) if it is differentiable of class \( C^p \) for every \( p \in \mathbb{N} \).

By Keller [2] the chain rule is valid for \( C^p_\alpha \), since \( \alpha \) is a finer limit structure than continuous convergence. From the universal property of continuous convergence follows that for any continuous map \( g : U \to L^k(E,F) \) the associated map \( \tilde{g} : U \times E^k \to F \) defined by \( \tilde{g}(x, h_1, \ldots, h_k) = g(x)(h_1, \ldots, h_k) \), \( x \in U \), \( h_i \in E \), is continuous. As the limit structure \( \alpha \) is always finer than \( \alpha \), we have that differentiability of class \( C^p_\alpha \) implies differentiability of class \( C^p_\alpha \). The latter is exactly the concept of differentiability used by Hamilton [1] to prove the inverse function theorem of Nash and Moser.

We first recall some definitions that will be needed.

Let \( E \) be a Fréchet space. A grading on \( E \) is an increasing sequence of norms \( (\| \cdot \|_r)_{r \in \mathbb{N}} \) on \( E \) which defines the topology on \( E \). Two gradings \( (\| \cdot \|_r)_{r \in \mathbb{N}} \) and \( (\| \cdot \|'_r)_{r \in \mathbb{N}} \) are equivalent if for some \( s \in \mathbb{N} \) \( \| x \|_r \leq c \| x \|'_{r+s} \) and \( \| x \|'_r \leq c \| x \|_r' \) for \( x \in E \), with a constant \( c \) which may depend on \( r \). A graded space is a Fréchet space together with an equivalence class of gradings. We say that a graded space \( E \) admits smoothing operators if we can find linear maps \( S_t : E \to E, \quad 1 \leq t < \infty \), such that for some \( r \) \( \| S_t(x) \|_{i+k} \leq c t^{r+k} \| x \|_i \) and \( \| S_t(x) - x \|_i \leq c t^{-r+k} \| x \|_{i+k} \) for all \( i, k \in \mathbb{N}, \quad 1 \leq t < \infty, \quad x \in E \) and some constant \( c \) which may depend on \( i \) and \( k \).

Let \( E \) and \( F \) be graded spaces and \( U \) open in \( E \). We say that a map \( f : U \to F \) is tame if for every \( x_0 \in U \) we can find a neighbourhood \( U_0 \) and a number \( r \in \mathbb{N} \) such that for every \( n \in \mathbb{N} \) the growth estimate \( \| f(x) \|_n \leq c (\| x \|_{n+r} + 1) \) for all \( x_0 \in U \), where the constant \( c \) may depend on \( n \).

In the proof of the inverse function theorem of Nash and Moser we shall also need the following result (Lemma 2, [3]): The composition of two continuous tame maps is continuous and tame.

**THEOREM.** Let \( E \) and \( F \) be graded spaces which admit smoothing operators. Let \( U \) be open in \( E \) and assume that
\[
(1) \quad f : U \to F \text{ is differentiable of class } C^\infty_\alpha \text{ and tame.}
\]
\[
(2) \quad D^k f : U \times E^k \to F \text{ is tame for every } k \in \mathbb{N}.
\]
\[
(3) \quad \text{For each } x \in U \text{ the derivative } \text{Df}(x) : E \to F \text{ is an isomorphism.}
\]
\[
(4) \quad \text{The map } Vf : U \to L^p_\alpha(F,E), \quad Vf(x) = (Df(x))^{-1}, \text{ is continuous.}
\]
\[
(5) \quad \text{Vf} : U \times F \to E \text{ is tame.}
\]
Then for any \( x_0 \in U \) we can find open neighbourhoods of \( x_0 \) and \( V_0 \) of \( f(x_0) \) such that \( f \) is a bijective map from \( U_0 \) onto \( V_0 \), and the inverse map \( f^{-1} : V_0 \to U_0 \) is differentiable of class \( C^{\infty}_a \), and the maps \( D^k f^{-1} : V_0 \times F^k \to E \) are tame for all \( k \in \mathbb{N} \). Furthermore, we have the formula \( D(f^{-1})(y) = Vf(f^{-1}(y)) \) for all \( y \in V_0 \).

**Proof.** The maps \( D^k f : U \times E^k \to F \) are continuous and tame, since \( f \) is differentiable of class \( C^\infty \) and assumption (2). Further, the assumptions (4) and (5) imply that also \( Vf : U \times F \to E \) is continuous and tame. Now we have that \( f \) is differentiable of class \( C^{\infty}_a \) and all \( D^k f \) are tame. \( Df(x) : E \to F \) is an isomorphism for every \( x \in U \) and the family of inverses \( Vf \) are continuous and tame maps. Consequently the conditions of the inverse function theorem of Nash-Moser are fulfilled (Theorem 1.1.1 p. 171 in [1]). Then for every \( x_0 \in U \) there exist neighbourhoods \( U_0 \) of \( x_0 \) and \( V_0 \) of \( f(x_0) \) such that \( f : U_0 \to V_0 \) is bijective and \( f^{-1} : V_0 \to U_0 \) is continuous and tame. Furthermore, the formula \( \lim_{t \to 0} f^{-1}(y + tw) - f^{-1}(y) = Vf(f^{-1}(y))w \) holds for all \( y \in V_0 \) and \( w \in F \), by the proof of Theorem 1.1.1 p. 186 in [1]. By induction on \( k \) we will prove the remaining part that \( f^{-1} : V_0 \to U_0 \) is differentiable of class \( C^{\infty}_a \) and \( D^{k+1} f^{-1} : V_0 \times F^k \to E \) is tame for every \( k \in \mathbb{N} \). From the formula \( Df^{-1} = Vf \cdot f^{-1} \) and assumption (4) follow that \( Df^{-1} : V_0 \to L(F_E) \) is continuous. Further we have that \( Df^{-1} : V \times F \to E \) is tame since \( Vf \) and \( f^{-1} \) are tame. Assume now it to be true for \( k \). From the definition of the \( \alpha \)-differentiability follows that the map \( f^{-1} \) is \( C^{k+1} \) if \( Df^{-1} \) is differentiable of class \( C^k \). Since \( Df^{-1} = Vf \cdot f^{-1} \), \( D^{k+1} f^{-1} \) is clearly tame so we only have to show that \( Vf \) is differentiable of class \( C^k \). By induction on \( p \). By Theorem 5.3.1, p. 102 in [1] we have that \( Vf \) is weakly differentiable and that \( D(Vf) : U_0 \times E \times F \to E \) is continuous and the formula \( [D(Vf)](x)(u, w) = -Vf(x)(D^2 f(x)(u, Vf(x)w)) \) holds for all \( x \in U_0 \), \( u \in E \) and \( w \in F \). Thus the derivative \( D(Vf) : U_0 \to L(E, F, E) \) can be factorized according to

\[
U_0 \xrightarrow{D^2 f, Vf} L_{\alpha}(E, F) \times L_{\alpha}(F, E) \xrightarrow{h} L_{\alpha}(E \times F, E),
\]

where \( h \) is defined by \( h(\phi, \psi) = -\psi \cdot \phi \cdot (id_E, \psi) \) for \( \phi = D^2 f(x) \) and \( \psi = Vf(x) \). By Theorem 0.3.5 in [2] \( h \) is continuous for \( \alpha = c \). Since the category \( K_{\alpha} \) is closed under finite products and \( ?_{\alpha} \) is a coreflector it follows that \( h \) is continuous. Thus it is true for \( p = 1 \). Since \( h \) is bilinear it is differentiable of class \( C^{\infty}_a \), and consequently the map \( Vf \) is differentiable of class \( C^{\infty}_a \) by induction. Thus the theorem is proved.

We shall now consider examples of coreflective subcategories of \( LC \) which are closed under finite products and the coreflectors \( ?_{\alpha} \) fulfill the assumption that the identity mapping \( C_{\alpha}(U, F) \to C_c(U, F) \) is continuous.

**Example 1.** Let \( K_{\alpha} \) be the category \( K_{\alpha} = LC ; ?_{\alpha} \) is the identity functor \( 1_{LC} = \alpha \).

**Example 2.** Let \( K_{\alpha} \) be the category \( K_{\alpha} \) of equable locally convex limit vector spaces [2]. The coreflector \( ?^e : LC \to K_{\alpha} \) is the identity on morphisms and on objects \( E \) it is characterized as follows: a filter \( F \) on \( E \) converges to zero in \( E \) iff \( \forall G \subseteq F \) for some filter \( G \) which converges to zero in \( E \).
EXAMPLE 3. Let $K$ be the category $K_M$ of Marinescu spaces [2]. The coreflector $\Phi^M : LC \to K_M$ is the identity on morphisms and on objects $E$ it is characterized as follows: a filter $F$ on $E$ converges to zero in $E^M$ iff $\forall G \in F$ and $\cap \{ KG : G \in G \} \in G$ for some filter $G$ which converges to zero in $E$.

EXAMPLE 4. Let $K_b$ be the category $K_b$ of bornological locally convex limit vector spaces. The coreflector $\Phi^b : LC \to K_b$ is the identity on morphisms and on objects $E$ it is characterized as follows: a filter $F$ on $E$ converges to zero in $E^b$ iff $\forall B \subseteq F$ for some bounded subset $B \subseteq E$, i.e. some set $B$ such that $\forall B$ converges to zero in $E$.

Example 1 gives us the inverse function theorem of Nash and Moser by Hamilton [1]. From example 3 we derive the inverse function theorem of Nash and Moser for the differentiability of class $C^\infty_M$ ($C^\infty_A$ in Keller [2]). In [4] Kriegl has discussed smooth mappings between locally convex spaces, where a mapping is called smooth iff its composition with smooth curves are smooth. He has compared this concept of smoothness with different $C^\infty_A$-differentiabilities (see [2]). From [2] and [4] follow that a mapping between Fréchet spaces is smooth iff it is $C^\infty_C$-differentiable. Thus the inverse function theorem of Nash and Moser is valid for this concept of smoothness.

REFERENCES

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