ON A GENERALIZATION OF THE CORONA PROBLEM

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ABSTRACT. Let $g, f_1, \ldots, f_m \in H^\omega(\Delta)$. We provide conditions on $f_1, \ldots, f_m$ in order that $|g(z)| \leq |f_1(z)| + \cdots + |f_m(z)|$, for all $z$ in $\Delta$, imply that $g$, or $g^2$, belong to the ideal generated by $f_1, \ldots, f_m$ in $H^\omega$.

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1. INTRODUCTION.

Let $H(\Delta)=H$ be the space of all holomorphic functions on $\Delta=\{z \in \mathbb{C} : |z|<1\}$, and let $H^\omega(\Delta)=H^\omega$ be the subspace of all bounded functions of $H(\Delta)$. Let $f_1, \ldots, f_m$ be functions in $H^\omega$ and let $g \in H^\omega$ satisfy the following condition:

$$|g(z)| \leq |f_1(z)| + \cdots + |f_m(z)| \quad \text{for all } z \in \Delta. \quad (1.1)$$

As a generalization of the corona problem (which was first solved by Carleson [1]) it is natural to ask if (1.1) implies that $g$ belongs to the ideal $I^\omega_H(f_1, \ldots, f_m)$ generated in $H^\omega$ by $f_1, \ldots, f_m$, i.e. if (1.1) implies the existence of $g_1, \ldots, g_m$ in $H^\omega$ such that

$$g = f_1 g_1 + \cdots + f_m g_m. \quad (1.2)$$

Rao, [2], has shown that the answer to this question is negative in general. On the other hand Wolff (see [3], th. 2.3) has proved that (1.1) implies that $g^2$ belongs to $I^\omega_H(f_1, \ldots, f_m)$. The question whether (1.1) implies the existence of $g_1, \ldots, g_m$ in $H^\omega$ such that

$$g^2 = f_1^2 g_1 + \cdots + f_m^2 g_m \quad (1.3)$$

is still open, as Garnett has pointed out ([4], problem 8.20).

In this work we obtain some results on this generalized corona problem, making use of techniques which appear in the theory of $A_p$ spaces, the spaces of entire functions with growth conditions introduced by Hörmander [5].

With the same aim of Berenstein and Taylor [6] in $A_p$, we introduce in $H^\omega$ the notion of jointly invertible functions (definition 3) and prove that if $f_1, \ldots, f_m$ are jointly invertible, condition (1.1) implies that $g$ belongs to $I^\omega_H(f_1, \ldots, f_m)$ (proposition 5). We also prove that if the ideal $I^\omega_H(f_1, \ldots, f_m)$ contains a weakly invertible
function having simple interpolating zeroes (see [3]), then again (1.1) implies that \( g \) belongs to \( I_{H}(f_{1}, \ldots, f_{m}) \) (theorem 6).

Finally, in the same spirit of Kelleher and Taylor [7] we introduce the notion of congeniality for \( m \)-tuples of functions in \( H^{m} \), and give a partial answer to the problem posed by Garnett ([4]): we prove that if \((f_{1}, \ldots, f_{m}) \in (H_{m})^{m}\) is congenial, then (1.1) implies \( g^{2} \in I_{H^{m}}(f_{1}, \ldots, f_{m}) \) (theorem 8).

2. WEAK INVERTIBILITY.

We first study some conditions under which (1.1) implies that \( g \in I_{H}(f_{1}, \ldots, f_{m}) \).

**DEFINITION 1.** A function \( f \) in \( H(A) \) is called weakly invertible if there exists a Blaschke product \( B \) such that \( f(z) = B(z) \) (\( z \) in \( A \)) with \( f \) invertible in \( H \).

The reason for this definition is the following simple criterion of divisibility for functions in \( H^{n} \).

**PROPOSITION 2.** Let \( f \in H^{n} \). Then \( f \) is weakly invertible if, and only if, for all \( g \in H^{n} \), the fact that \( g/f \in H^{n} \) implies \( g/f \in I_{H} \).

**PROOF.** Suppose \( f \) is weakly invertible: then there exists a Blaschke product \( B \) such that \( f(z) = B(z) \) with \( f \) invertible in \( H \). Since \( g/f \) is holomorphic and since \( B \) contains exactly the zeroes of \( f \), it follows that \( g/B \in H \); however, since \( B \) is a Blaschke product, \( g/B \) implies, [8], that \( g/B \in H \). Since \( I/ = \{ \} \) one has \( g/f = (g/B)(1/\), i.e. \( g/f \in H^{n} \). Conversely, suppose that for all \( g \in H^{n} \), the fact that \( g/f \in H^{n} \) implies \( g/f \in H^{n} \).

An extension of the notion of weak invertibility to \( m \)-tuples of functions in \( H^{m} \) is given by the following definition, analogous to the one given by Berenstein and Taylor for the spaces \( A \) in [6].

**DEFINITION 3.** The functions \( f_{1}, \ldots, f_{m} \in H^{m} \) are called jointly invertible if the ideal generated by \( f_{1}, \ldots, f_{m} \) in \( H^{m} \) coincides with \( I_{H}(f_{1}, \ldots, f_{m}) \) for any \( z \in \Delta \), there exists a neighborhood \( U \) of \( z \) and \( \lambda_{1}, \ldots, \lambda_{m} \) in \( H(A) \) such that \( g = \lambda_{1} f_{1} + \ldots + \lambda_{m} f_{m} \) on \( U \).

In view of Cartan's theorem B, it follows immediately that \( f_{1}, \ldots, f_{m} \) are jointly invertible if, and only if, \( I_{H}(f_{1}, \ldots, f_{m}) = I_{H}(f_{1}, \ldots, f_{m}) \), the latter being the ideal generated by \( f_{1}, \ldots, f_{m} \) in \( H(\Delta) \). As a consequence of the corona theorem, one has.

**PROPOSITION 4.** Let \( b \in H^{m} \) be weakly invertible, and let \( f_{1}(z) = b(z)f_{1}(z), \ldots, f_{m}(z) = b(z)f_{m}(z) \), for \( f_{1}, \ldots, f_{m} \) in \( H^{m} \) such that \( |f_{1}(z)| + \ldots + |f_{m}(z)| \geq \delta \) for some \( \delta > 0 \) and all \( z \) in \( \Delta \). Then \( f_{1}, \ldots, f_{m} \) are jointly invertible.

**PROOF.** Let \( g \in H^{m} \) belong to \( I_{H}(f_{1}, \ldots, f_{m}) \). There exist \( \lambda_{1}, \ldots, \lambda_{m} \) in \( H(\Delta) \) such that

\[
g(z) = \lambda_{1} f_{1}(z) + \ldots + \lambda_{m} f_{m}(z) \quad \text{(all } z \in \Delta) \tag{2.1}\]

i.e., for all \( z \) in \( \Delta \),

\[
g(z) = b(z)[\lambda_{1}(z)f_{1}(z) + \ldots + \lambda_{m}(z)f_{m}(z)]. \tag{2.2}\]

Since \( b \) is invertible, and \( g/b \in H^{m} \), it follows that \( g/b - f_{1} + \ldots + f_{m} f_{m} \). By the corona theorem, then, it follows that there are \( h_{1}, \ldots, h_{m} \) in \( H^{m} \) such that

\[
g(z) = h_{1}(z)f_{1}(z) + \ldots + h_{m}(z)f_{m}(z), \tag{2.3}\]

therefore

\[
g(z) = (g(z)b(z) = h_{1}(z)f_{1}(z) + \ldots + h_{m}(z)f_{m}(z) \tag{2.4}\]
and the assertion is proved.

Let now \( f_1, \ldots, f_m, g \in \mathbb{H}^\omega(\Delta) \), and suppose that (1.1) holds. It is well known, [2], that in general (1.1) does not imply that \( g \in \mathbb{I}_n^\omega(f_1, \ldots, f_m) \). However, (1.1) certainly implies that \( g \in \mathbb{I}_{n}^\omega(f_1, \ldots, f_m) \) and hence

**Proposition 5.** Let \( f_1, \ldots, f_m \) be jointly invertible. Then if \( g \) satisfies condition (1.1), it follows that \( g \in \mathbb{I}_n^\omega(f_1, \ldots, f_m) \).

A different situation in which (1.1) implies that \( g \in \mathbb{I}_n^\omega(f_1, \ldots, f_m) \) occurs when at least one of the \( f_j \)'s, say \( f_1 \), is weakly invertible and has simple zeroes which form an interpolating sequence ([3]); this happens, for example, when \( f_1 \) is an interpolating Blaschke product with simple zeroes ([3]). Indeed, following an analogous result proved in [7] for the space of entire functions of exponential type, one has:

**Theorem 6.** Let \( f_1, \ldots, f_m \in \mathbb{H}^\omega \), and suppose \( f_1 \) is weakly invertible with simple, interpolating zeroes. Then if \( g \in \mathbb{H}^\omega \) satisfies condition (1.1) it follows that \( g \) belongs to \( \mathbb{I}_{n}^\omega(f_1, \ldots, f_m) \).

**Proof.** Choose \( a_{i,j} \in \mathbb{C}, i=2, \ldots, m, j \geq 1 \), such that for \( \{z_j\} = \{z \in \Delta : f_1(z) = 0\} \) it is \( |a_{i,j}| = 1 \) and \( a_{i,j} f_1(z_j) \neq 0 \). Define now \( b_{i,j} \in \mathbb{C} \) (i, j as before) by

\[
 b_{i,j} = \begin{cases} 
 0 & \text{if } f_2(z_j) = \ldots = f_m(z_j) = 0 \\
 a_{i,j} g(z_j)/(|f_2(z_j)|^2 + \ldots + |f_m(z_j)|^2) & \text{otherwise.} 
\end{cases}
\]

By (1.1) it follows \( |b_{i,j}| \leq 1 \) (all \( i,j \)), and since \( \{z_j\} \) is interpolating, one finds \( h_2, \ldots, h_m \in \mathbb{H}^\omega \) such that \( h_j(z_j) = b_{i,j} \). Therefore the function \( h = g - (h_2 f_2 + \ldots + h_m f_m) \) belongs to \( \mathbb{H}^\omega \) and vanishes at each \( z_j \). The simplicity of the zeroes of \( f_1 \) shows that \( f_1/h \in \mathbb{H}^\omega \), and the invertibility of \( f_1 \) implies \( h/f_1 \in \mathbb{H}^\omega \). The thesis now follows, since \( g = f_1 h_1 + \ldots + f_m h_m \).

It is worthwhile noticing that the hypotheses of Proposition 5 and Theorem 6 are not comparable. Consider, indeed, the following conditions on \( f_1, \ldots, f_m \in \mathbb{H}^\omega \):

(C1) \( f_1, \ldots, f_m \) are jointly invertible.

(C2) there exists \( j (1 \leq j \leq m) \) such that \( f_j \) is invertible, with an interpolating sequence of zeroes, all of which are simple.

Then (C1) does not imply (C2): take \( m = 1 \) and \( f_1 \) weakly invertible with non-simple zeroes.

On the other hand, also (C2) does not imply (C1): consider \( f_1 \in \mathbb{H}^\omega \) with simple interpolating zeroes \( \{z_n\} \); let \( f_2 \in \mathbb{H}^\omega \) be a function such that \( f_2(z_n) = 1/n \) (such a function certainly exists since \( \{z_n\} \) is an interpolating sequence); now \( f_1 \) and \( f_2 \) have no common zeroes, and hence \( \mathbb{I}_{\infty}^\omega(f_1, f_2) \); however \( \mathbb{I}_{\infty}^\omega(f_1, f_2) \) since if \( = \lambda_1 f_1 + \lambda_2 f_2 \), then it is \( \lambda_2(z_n) = n, i.e. \lambda_2 \in \mathbb{H}^\omega \). Therefore the pair \( f_1, f_2 \) satisfies (C2) but not (C1).

3. CONGENIALITY.

In this section we describe a class of \( m \)-tuples of functions in \( \mathbb{H}^\omega(\Delta) \), for which condition (1.1) implies that \( g \in \mathbb{I}_n^\omega(f_1, \ldots, f_m) \).

**Definition 7.** An \( m \)-tuple \( (f_1, \ldots, f_m) \) of functions in \( \mathbb{H}^\omega \) is called congenial if, for all \( i,j=1, \ldots, m \),

\[
\left( f_i f_j - f_j f_i \right) / \left| f_i \right|^2 \left| f_j \right|^2 \text{ belongs to } \mathbb{L}^\omega(\Delta),
\]

where \( \left| f(z) \right|^2 = \sum_{i=1}^{m} \left| f_i(z) \right|^2 \), \( \left| f'(z) \right|^2 = \sum_{i=1}^{m} \left| f_i'(z) \right|^2 \), and \( f'_i = \partial f_i / \partial z \).
Notice that the class of congenial m-tuples is not empty. Indeed, one might consider pairs \( f_1, f_2 \) in \( \mathbb{H}^m \) which, at their common zeroes, satisfy some simple conditions on their vanishing order easily deducible from Definition 7. For example, one can ask that 
\[ f_1(z_0) = f_2(z_0) = 0, \quad f_1'(z_0) \neq 0, \quad f_2'(z_0) = 0. \]
As a partial answer to problem 8.20 in [4], we prove the following

**THEOREM 8.** Let \( f_1, \ldots, f_m, g \in \mathbb{H}^m(\Delta) \), and suppose \((f_1, \ldots, f_m)\) be congenial. If \( g \) satisfies (1.1), then \( g \in \mathbb{I}_m^\infty(f_1, \ldots, f_m) \), i.e. there are \( g_1, \ldots, g_m \) in \( \mathbb{H}^m \) such that (on \( \Delta \))

\[
g^2(z) = f_1(z)g_1(z) + \ldots + f_m(z)g_m(z) \quad (3.1)
\]

**PROOF.** We mainly follow the proof due to Wolff, [3], of the fact that (1.1) implies that \( g \in \mathbb{I}_m^\infty \). We can assume \( |f_j| \leq 1, \quad |g_j| \leq 1 \), and \( f_j, g_j \in \mathbb{H}(\overline{\Delta}) \) \((j = 1, \ldots, m)\). Put \( \psi_j = \frac{\overline{f_j}}{|f|} \)

\[
\psi_j \text{ is bounded and } C^\infty \text{ on } \overline{\Delta}
\]

and consider the differential equation

\[
\partial_{\overline{z}} \psi_j = \psi_j \partial_{\overline{z}} \psi_j = g^2 \Gamma_j, k \quad (1 \leq j, k \leq m)
\]

for \( \Gamma_j, k \). If solutions \( \psi_j \) exist, then clearly \( g^2 \in \mathbb{I}_m^\infty \) and (3.1) holds (indeed \( g_j = 0 \) and \( g_j \) is bounded on \( \Delta \)). In order to prove that (3.2) admits a solution in \( \mathbb{I}_m^\infty \), it is enough to show that \( |g^2 \Gamma_j, k|^2 \log(1/|z|) \) dy dx and \( g \Gamma_j, k / \partial z \) are Carleson measures for \( 1 \leq j, k \leq m \).

As far as \( |g^2 \Gamma_j, k|^2 \log(1/|z|) \) dy dx is concerned, notice that, by the congeniality of \( (f_1, \ldots, f_m) \), it is

\[
|g^2 \Gamma_j, k|^2 \leq |g|^{4} |f_j|^2 \left| \sum_k f_k (\overline{f_k} \delta_{j,k} - f_k f_j) \right|^2 / |f|^{6} \leq C |f^*|^2.
\]

On the other hand,

\[
3 \Gamma_j, k / \partial z = 2g \Gamma_j, k + g^2 \Gamma_j, k / \partial z;
\]

again by the congeniality of \( (f_1, \ldots, f_m) \), one has

\[
|g \Gamma_j, k| \leq |g| |g^*| |f_j| \left| \sum_k f_k (\overline{f_k} \delta_{j,k} - f_k f_j) \right| / |f|^{6} \leq C \left( |g^*|^2 + |f^*|^2 / |f| \right),
\]

and

\[
|g^2 \Gamma_j, k / \partial z| = |g| \left| f_j \left| \sum_k f_k (\overline{f_k} \delta_{j,k} - f_k f_j) / |f| \right|^2 \left| \sum_k f_k (\overline{f_k} \delta_{j,k} - f_k f_j) / |f| \right|^4 \right| / |f|^{6} \leq C \sum_k |f_k|^2 / |f|^{2}.
\]

This concludes the proof.

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