RECURRENT AND WEAKLY RECURRENT POINTS IN $\beta G$

MOSTAFA NASSAR

Department of Mathematics
Acadia University
Wolfville, N.S. BOP 1XO Canada

(Received April 19, 1983)

ABSTRACT. It is shown in this paper that if $\beta G$ is the Stone-Čech compactification of a

group $G$, and $G$ satisfying a certain condition, then there is a weakly recurrent point

in $\beta G$ which is not almost periodic, and if another condition will be added, then there

is a recurrent point in $\beta G$ which is not almost periodic point.

KEY WORDS AND PHRASES. Topological group, recurrent point, StoneČech Compactification,

almost periodic point.

1980 AMS SUBJECT CLASSIFICATION CODE. 43A60.

1. INTRODUCTION.

Let $G$ be infinite group denoted by $B(G)$ the spaces of all bounded real-valued

functions with the usual sup norm, and by $B(G)^*$ it’s conjugate. An $g$-mean is a function

$\phi' \in B(G)^*$ such that $\| \phi' \| = 1$, $\phi'(u) = 1$ where $u$ is the unit function, i.e. $u(g) = 1$ for

all $g \in G$, $\phi' (f) = \phi'(f)$ for all $f \in B(G)$ where $f(s) = f(gs)$, $s \in G$, and $\phi'(f) \geq 0$ if

$f \geq 0$. If such $g$-mean exists we call $G$ amenable group.

If $G$ is amenable group with the discrete topology, $G$ be discrete set, as completely

regular topological space $G$ has a Stone-Čech Compactification $\beta G$. In W. Rudin [1] the

space of real-valued continuous functions on $\beta G$ and the space of bounded real-valued

functions on $G$ with the usual sup norm are isomorphic as Banach spaces. Any $g$-mean $\phi'$

as a functional on $C(\beta G)$ is represented by Riesz representation theorem as a measure $\phi$

defined on the Borel sets of $\beta G$. The correspondence being characterized by

$\phi'(f) = \int_{\beta G} f d\phi.$

For any $g \in G$ we have a continuous mapping $\tilde{g}$ of $G$ into $\beta G$ defined by $\tilde{g}(g_1) = gg_1$,

$g_1 \in G$, $\tilde{g}$ has a unique continuous extension to $\beta G$, the extension mapping will also be

denoted by $\tilde{g}$. If a subset of $G$ is any subset denote by $\tilde{A}$ the open-closed subset of

$\beta G \setminus G = \tilde{G}$ obtained as $G \cap \tilde{A}$, where $\tilde{A}$ is the closure of $A$ in $\beta G$. If $G$ is infinite left

cancellation semigroup, then for $s \in G$ and $B$ subset of $G$, $\tilde{s}B = (sB)$ Chou [2], $\tilde{g}$ is a

homeomorphism of the compact Hausdorff space $G$ onto itself denote by $\tilde{M}$ the set of all

$g$-invariant probability measures on $\beta G$, and the upper density of a subset $A$ of $G$ by

$\underline{d}_{\tilde{g}} (A) = \sup \{ \mu(A) : \mu \in \tilde{M} \}.$

2. THIN AND STRONGLY DISCRETE POINT.

DEFINITION 2.1. A subset $A$ of $G$ is said to be thin if $g_1 A \cap g_2 A$ is finite subset of

$G$ for each pair of distinct elements $g_1, g_2 \in G$. 

DEFINITION 2.2. \( \omega \in \mathbb{BG} \setminus \mathbb{G} \) is said to be discrete if the orbit of \( \omega \), \( \text{O}(\omega) = \{ g \omega \mid g \in \mathbb{G} \} \) is discrete with respect to the relative topology that is if and only if there exists a neighborhood \( U \) of \( \omega \) such that \( g \omega U \) if \( g \neq e \). Denote by \( D^\mathbb{G} \) the set of all discrete points in \( \mathbb{G} \).

DEFINITION 2.3. \( \omega \in \mathbb{BG} \setminus \mathbb{G} \) is said to be strongly discrete if there exists a neighborhood of \( \omega \) such that \( g_1 \cup g_2 U = \emptyset \) if \( g_1 \neq g_2 \). Denote by \( SD^\mathbb{G} \) the set of all strongly discrete points in \( \mathbb{G} \).

REMARK. \( SD^\mathbb{G} \) is a subset of \( D^\mathbb{G} \). For take \( g_1 = e \) the unit element in \( \mathbb{G} \), \( g_2 = g \neq e \) so \( \omega \in SD^\mathbb{G} \) implies there exists a neighborhood \( U \) of \( \omega \) such that \( U g_1 U \) implies \( g \omega U \) implies \( \omega \in D^\mathbb{G} \).

DEFINITION 2.4. A point \( \omega \in \mathbb{BG} \setminus \mathbb{G} \) is said to be almost periodic if for every neighborhood \( U \) of \( \omega \) there is a subset \( A \) of \( \hat{\mathbb{G}} \) which satisfy: (i) \( \text{Am} \) is a subset of \( U \), (ii) there exists a finite subset \( K \) of \( \mathbb{G} \) such that \( \text{GKA} \) or equivalently for each neighborhood \( U \) of \( \omega \) the set \( A = \{ g \in \mathbb{G} \mid g \omega U \} \) is relatively dense, in the sense there exists \( g_1, g_2, \ldots, g_n \in \mathbb{G} \) such that \( g_1^u g_2^u g_3^u \ldots g_n^u A = \mathbb{G} \). Denote by \( A^\mathbb{G} \) the set of all almost periodic points in \( \mathbb{BG} \).

PROPOSITION 2.5. \( D^\mathbb{G} \cap A^\mathbb{G} = \emptyset \)

PROOF. If \( \omega \in D^\mathbb{G} \), then there is a neighborhood \( V \) of \( \omega \) in \( \mathbb{BG} \) such that \( V \cap o(\omega) = \{ \omega \} \), hence \( \omega \) is not almost periodic point otherwise there exists a subset \( A \) of \( \mathbb{G} \) such that \( \text{Am} \) is a subset of \( V \) which is a contradiction to the conclusion \( V \cap o(\omega) = \{ \omega \} \). Then \( \omega \notin A^\mathbb{G} \) and so \( D^\mathbb{G} \cap A^\mathbb{G} = \emptyset \).

REMARK. If \( A \) is a subset of \( \hat{\mathbb{C}} \), \( A \) is empty if and only if \( A \) is finite, also \( g \in (gA)^{\hat{\mathbb{G}}} \) for \( g \in \mathbb{G} \).

THEOREM 2.6. (1) If \( A \) is a thin subset of the group \( \mathbb{G} \) then \( \hat{d}(A) = 0 \). (2) \( SD^\mathbb{G} = U(A : A \text{ is a thin subset of } \mathbb{G}) \).

PROOF. (1) Suppose that \( A \) is thin so \( g_1^u g_2^u A \) is finite for each distinct pair of elements \( g_1, g_2 \in \mathbb{G} \). But

\[
\text{cl}(g_1^u g_2^u A)^{\hat{\mathbb{G}}} = (\text{cl}_{\hat{\mathbb{G}}} g_1^u A \cap \text{cl}_{\hat{\mathbb{G}}} g_2^u A)^{\hat{\mathbb{G}}} = (\text{cl}_{\hat{\mathbb{G}}} g_1^u A)^{\hat{\mathbb{G}}} \cap (\text{cl}_{\hat{\mathbb{G}}} g_2^u A)^{\hat{\mathbb{G}}} = (g_1^u A)^{\hat{\mathbb{G}}} \cap (g_2^u A)^{\hat{\mathbb{G}}} = g_1^u A \cap g_2^u A.
\]

If \( A \) is thin and \( \phi \in \hat{\mathbb{M}} \) the set of all invariant probability measures on \( \hat{\mathbb{G}} \). So \( \phi^{\hat{\mathbb{G}}} = 1 \), hence for any distinct elements \( g_1, g_2, \ldots, g_n \in \mathbb{G} \), \( g_1, g_2, \ldots, g_n \in \mathbb{G} \) are distinct and

\[
1 = \phi^{\hat{\mathbb{G}}} = \phi(g_1^u A) = \sum_{i=1}^{n} \phi(g_1^u A) = n \phi(g_1^u A) \implies \phi(g_1^u A) = 1.
\]

(2) \( SD^\mathbb{G} = \{ \omega \in \mathbb{G} : \text{There exists neighborhood } U \text{ of } \omega, g_1^u g_2^u U = \emptyset \text{ for } g_1 \neq g_2 \} \)

\[
= \{ \omega \in \mathbb{G} : \text{There exists neighborhood } U \text{ of } \omega, g_1^u g_2^u U = \emptyset \text{ for } g_1 \neq g_2 \}
\]

\[
= U(\text{cl}_{\hat{\mathbb{G}}} g_1^u g_2^u A = \emptyset \text{ for all distinct pair of elements } g_1, g_2 \in \mathbb{G})
\]

\[
= U(\text{cl}_{\hat{\mathbb{G}}} g_1^u g_2^u A \text{ is finite})
\]

\[
= U(A : A \text{ is a thin subset of } \mathbb{G}).
\]

3. WEAKLY RECURRENT AND RECURRENT POINTS.

DEFINITION 3.1. \( \omega \in \mathbb{BG} \) is said to be \( \hat{\mathbb{G}} \)-recurrent point if, for each neighborhood \( V \) of \( \omega \) the set \( \{ n \in \mathbb{N} : \omega \in V \} \) is infinite. Denote by \( R^{\hat{\mathbb{G}}} \) the set of all \( \hat{\mathbb{G}} \)-recurrent points, and by \( R^\mathbb{G} \) = the complement of \( D^\mathbb{G} \) in \( \mathbb{G} \), to be the set of all recurrent points. So

\[
R^\mathbb{G} \supseteq U_{g \in \mathbb{G}} R^{\hat{\mathbb{G}}}.
\]
DEFINITION 3.2. Denote by $WR_G^*$ the set of all weakly recurrent points in $G^*$, it is the complement of $SD_G^*$ in $G^*$.

Since $D^G_G \cap A^G = \emptyset$, proposition 2.5 which implies $A^G \subseteq R_G^*$. We call a subset $A$, a $C$-subset of $G$ provided that

(i) $\hat{d}(A) > 0$

(ii) $\hat{d}(K^{-1}A) < 1$ for every finite subset $K$ of $G$, equivalently.

REMARK. $C$ stands for Chou. Denote by $AC$ the class of all amenable semigroup which has a $C$-subset. This class contains the semigroup $N$ of positive integers, the group $Z$ of integers, all countably infinite locally finite groups, all infinite abelian cancellation semigroups, and all infinite solvable groups, with the discrete topology for more details see Fairchild [3].

One reason for studying the $C$-subset is the following result.

PROPOSITION 3.4. Suppose $G$ contains a $C$-subset $A$ then $A \cap A^G = \emptyset$.

PROOF. Suppose $\hat{A} \cap \hat{A}^G = \emptyset$, since $\hat{A}$ is open subset contains $\omega$.

Let $B = \{ g \in G : g \omega \in \hat{A} \}$ so there exists a finite subset $K$ of $G$ such that $G = B \cup K^G = \{ g \in K^G : g \omega \in (K^{-1}A) \}$ implies $\hat{A} \cap \hat{G} = \emptyset$. But $\hat{G}$ is closed invariant set implies there exists $\phi$ a probability measure such that $\text{supp} \phi \subseteq (K^{-1}A)$ implies $\phi'(1_{K^{-1}A}) = 1$ which contradicts the definition of $C$-subset.

Then $\hat{A} \cap \hat{A}^G = \emptyset$.

REMARK. If $A$ is a subset of $G$, $I_A$ denote the function $1$ on $A$ and $0$ otherwise.

THEOREM 3.5. If $G \in AC$ then there exists a weakly recurrent point in $BG$ which is not almost periodic, in other words $A^G \notin WR_G^*$.

PROOF. Theorem 2.6 shows that $SD_G^* = \{ A : A^G \text{ is a thin subset of } G \}$, but $d(A) > 0$ where $A^*$ is a $C$-subset of $G$, then $A$ is not thin subset implies $A^G \notin SD_G^*$, so $\hat{A} \cap WR_G^* = \emptyset$. In Proposition 3.4 we proved that if $A$ is a $C$-subset then $\hat{A}^G \cap \hat{A} = \emptyset$, hence we get $A^G \notin WR_G^*$. So there exists a weakly recurrent point in $BG$ which is not almost periodic. Moreover $A^G \cup SD_G^* = \emptyset$.

The only known method to find $\hat{g}$-recurrent points is to apply Zorn's lemma to find a $\hat{g}$-minimal set $K$, then show that each $w \in K$ is $\hat{g}$-almost periodic and therefore $\hat{g}$-recurrent.

In theorem 3.8 we are going to produce many other $\hat{g}$-recurrent points for a reasonable class of semigroups.

Chou [4] has proved that

THEOREM (Chou): Let $\phi$ be a homeomorphism of a compact Hausdorff space $X$ onto itself. Suppose that $T_1 \supset T_2 \supset \ldots$ is a sequence of non-empty closed subsets of $X$ such that a sequence of positive integers $k_1 < k_2 < \ldots$ can be found to satisfy $\phi_{n+1} T_n \supset T_n$.

Then $\bigcap_{n=1}^{\infty} T_n$ contains a $\phi$-recurrent point.

LEMMA 3.6. Suppose that $A$ is a subset of $G$, $\hat{d}_G^*(A) > 0$, and $n \in \mathbb{N}$. Then there exists a subset of $B$ of $A$, $s \in \mathbb{N}$, $s \in \mathbb{N}$ such that $\hat{d}_G^*(B) > 0$ and $\hat{G}^n B \subseteq A$.

PROOF. By definition of upper $\hat{g}$-density, there exists $\mu \in \mathfrak{M}^G$ such that $\mu(A^*) > 0$. If for each $s \in \mathbb{N}$, $\mu(A^n \hat{G}^s A^*) = 0$. Then
322 MOSTAFA NASSAR

\[ \sum_{i=0}^{\infty} \mu(\gamma^{-in}A) = \mu(\bigcup_{i=0}^{\infty} \gamma^{-in}A) \leq 1 \]

This contradicts the fact that \( \mu \) is a \( \gamma \)-variant \( (\mu(\gamma A) = \mu(\gamma^{-1}A)) \).

Therefore there exists some \( n \) such that \( \mu(\gamma A) > 0 \). But since \( \gamma A = (gA)^{\gamma} \) so

\( (\gamma A)^{\gamma} = \gamma (g A)^{\gamma} = (A^{\gamma} g)^{\gamma} \). Take \( B = A \cap (^{\gamma} g A) \) then \( \mu(B) > 0 \) and \( g^{\gamma} B \subseteq A \), but \( \mu(N) \)

so \( \mu(B) > 0 \).

DEFINITION 3.7. The group \( G \) is said to be nontorsion group if \( G \) contains an element of infinite order.

THEOREM 3.8. If \( G \) is nontorsion group, \( G \subseteq A \), then there is a recurrent point in \( B G \) which is not almost periodic. In other words \( A_G \subseteq R_G \).

PROOF. Since \( G \) has a C-subset we may assume that \( A \) to be a C-subset hence by

proposition 3.4 \( \gamma A \subseteq G = \phi \). Therefore it remains to produce a recurrent point in \( A \).

By lemma 3.6 it is easy to construct \( s_1 < s_2 < ... \) and \( A = A_1 \supseteq A_2 \supseteq ... \), inductively, such that \( d^\gamma(A_1) > 0 \) and \( \gamma^{\gamma-1} A_1 \subseteq A \subseteq A_1 \), \( i = 1, 2, 3 ... \) therefore \( A \) contains a recurrent point by applying Chou's theorem to the case \( \phi = \gamma, X = G, k_1 = s_1 \) and \( T_n = A_n \) noting that \( g \) is an element of \( G \) of infinite order, so the function \( \gamma \) is nonperiodic and hence there is a recurrent point which is not almost periodic, and since \( A_G \subseteq R_G \) we get \( A_G \subseteq R_G \).

In fact \( R_G \) is much bigger than \( A_G \).

The above theorem tells us that \( A \subseteq A_G \subseteq R_G \), this answers the question raised by

Nilsen [5].

CONJECTURE. If \( G \) is amenable group then there is a recurrent point in \( B G \) which is not almost periodic point. In other words: \( A_G \subseteq R_G \).

ACKNOWLEDGEMENT. I would like to thank Professor Ching Chou at State University of New York at Buffalo for his encouragement and advice.

REFERENCES

4. CHOU, C. Minimal Sets, Recurrent Points and Discrete Orbits on \( B \mathbb{N} \setminus \mathbb{N} \), Illinois, J., of Math. 22 (1978), 54-63.
5. NILLSSEN, R. Discrete Orbits in \( B \mathbb{N} \setminus \mathbb{N} \), Colloq. Math. 33 (1975), 71-81.