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ORDER COMPATIBILITY FOR CAUCHY SPACES AND CONVERGENCE SPACES

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ABSTRACT. A Cauchy structure and a preorder on the same set are said to be compatible if both arise from the same quasi-uniform convergence structure on X. However, there are two natural ways to derive the former structures from the latter, leading to "strong" and "weak" notions of order compatibility for Cauchy spaces. These in turn lead to characterizations of strong and weak order compatibility for convergence spaces.

KEY WORDS AND PHRASES. *Preordered Cauchy space, preordered convergence space, weakly preordered space, quasi-uniform convergence space.*

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INTRODUCTION

L. Nachbin, [3], introduced order compatibility between a uniform structure and a preorder $\mathcal{O}$ on the same set X by requiring the existence of a quasi-uniformity $\mathcal{S}$ on X such that $\mathcal{U} = \mathcal{S} \cup \mathcal{S}^{-1}$ and $\mathcal{O} = \cap \mathcal{S}$. An analogous procedure was used by the authors in an earlier paper, [2], to define order compatibility between a Cauchy structure $\mathcal{C}$ and a partial order $\mathcal{O}$ on X; the principal deviation from Nachbin's definition is the replacement of "quasi-uniformity" by "quasi-uniform convergence structure". However we required in [2] that the quasi-uniform convergent structure employed be coarser than that generated by the order $\mathcal{O}$. If the latter restriction is removed, a weaker form of order compatibility is obtained.

This paper examines both types of order compatibility for Cauchy spaces. The "strong" version is the subject of Section 1; here we extend the results of [2] by assuming that $\mathcal{O}$ is a preorder rather than a partial order. A convergence space derived from a "preordered Cauchy space" is called a "preordered convergence space", and these are shown to be precisely the locally convex convergence spaces for which the preorder is closed.

In Section 2, we consider the "weak" form of order compatibility, a topic not previously studied. The characterization of "weakly preordered Cauchy spaces", is given in two forms, the first of which provides a convenient comparison with the

preordered Cauchy spaces of Section 1. The second leads to a simple characterization of weakly preordered convergence spaces.

The terminology and notation of [2] will be used in this paper without further reference. However we shall always assume in this present paper that $\mathcal{O}$ is a preorder (i.e., a transitive, reflexive relation).

1. PREORDERED CAUCHY SPACES (STRONG COMPATIBILITY).

From Definitions 2.1 and 2.2 of [2], we obtain the definitions of preordered uniform convergence space and preordered Cauchy space by simply assuming that $\mathcal{O}$ is a preorder instead of a partial order. The latter term will be abbreviated "p.c.s". A p.c.s. $(X, \mathcal{O}, \mathcal{C})$ for which $\mathcal{O}$ is a partial order is called an ordered Cauchy space (abbreviated "o.c.s.") In a p.c.s. $(X, \mathcal{O}, \mathcal{C})$, we have "strong" compatibility between $\mathcal{O}$ and $\mathcal{C}$ because of the requirement in Definition 2.1, [2], that the filter $\langle\mathcal{O}\rangle$ on $X \times X$ generated by $\mathcal{O}$ be in σ. The relaxation of this requirement leads to the "weak" compatibility studied in the next section.

In this section, we consider triples of the form $(X, \mathcal{O}, \mathcal{C})$, where $(X, \mathcal{O})$ is a preordered set and $\mathcal{C}$ a Cauchy structure on X. In determining when $(X, \mathcal{O}, \mathcal{C})$ is a p.c.s., the preorder $\lesssim$ on the set $\mathcal{C}$ of Cauchy filters, as defined on p. 486, [2], plays a vital role since it is required in formulating two of the following conditions (see also p. 487, [2]):

$(OC)_1$ $\quad \mathcal{F}^\wedge \in \mathcal{C}$ whenever $\mathcal{F} \in \mathcal{C}$;

$(OC)_2$ $\quad$ If $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, $\mathcal{F} \lesssim \mathcal{G}$, and $\mathcal{G} \lesssim \mathcal{F}$, then $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$;

$(OC)_3$ $\quad \dot{x} \lesssim \dot{y}$ implies $x \leq y$.

The first condition defines "local convexity" for a Cauchy structure on a preordered set; recall that $\mathcal{F}^\wedge = \mathcal{O}(\mathcal{F}) \cup \mathcal{O}^{-1}(\mathcal{F})$ is the convex hull of $\mathcal{F}$. The second condition asserts that the preorder $\lesssim$ on $\mathcal{C}$ is antisymmetric relative to Cauchy equivalence classes, and the third turns out to be equivalent to the order $\mathcal{O}$ being closed in $X \times X$.

It was shown in Theorem 2.9, [2], that when $\mathcal{O}$ is a partial order, $(X, \mathcal{O}, \mathcal{C})$ is an o.c.s. iff conditions $(OC)_1$, (OC_2), and $(OC)_3$ are all satisfied. In this section we show that when $\mathcal{O}$ is a preorder, $(X, \mathcal{O}, \mathcal{C})$ is a p.c.s. iff the same conditions hold. This task is made easier by the fact that all lemmas and Propositions in Section 2, [2] prior to Theorem 2.9 remain valid under the assumption that $\mathcal{O}$ is a preorder rather than a partial order. Whereas the statement of Theorem 2.9, [2] remains valid when $\mathcal{O}$ is a preorder, the proof of this theorem must be altered, since it makes explicit use of the assumption that $\mathcal{O}$ is antisymmetric. Since, in particular Proposition 2.8, [2] is valid when $\mathcal{O}$ is a preorder, the principal theorem of this section reduces to showing that $(OC)_1$, $(OC)_2$, and $(OC)_3$ imply that $(X, \mathcal{O}, \mathcal{C})$ is a p.c.s. Thus, for the remainder of this section, we assume that $(X, \mathcal{O}, \mathcal{C})$ is a preordered set with Cauchy structure which satisfies $(OC)_1$, $(OC)_2$, and $(OC)_3$.

Given $(X, \mathcal{O}, \mathcal{C})$, an equivalence relation on the elements of X is defined as follows: $x \sim y$ iff $(x,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$. Let $E_x = \{y \in X : x \sim y\}$ be the equivalence class containing x, and let $\langle E_x \rangle$ be the filter of oversets of E_x.

LEMMA 1.1. For each $x \in X$, $\langle E_x \rangle \to y$ in $(X, q_{\mathcal{C}})$ for all $y \in E_x$. In particular, $\langle E_x \rangle \in \mathcal{C}$.

PROOF. It is obvious that $E_y = \mathcal{O}(y) \cap \mathcal{O}^{-1}(y)$, for all $y \in X$. Since $E_x = E_y$ and $\dot{y} \to y$, the conclusion follows by $(OC)_1$. $\square$

LEMMA 1.2. If $\mathcal{F}$ is a filter on X such that $\langle \mathcal{O} \rangle \subseteq \mathcal{F} \times \mathcal{F}$, then there is $x \in X$ such that $E_x \in \mathcal{F}$. Furthermore, $\mathcal{F} \in \mathcal{C}$ and $\mathcal{F} \to y$ in $(X, q_{\mathcal{C}})$ for all $y \in E_x$.

PROOF. Choose $F \in \mathcal{F}$ such that $F \times F \subseteq \mathcal{O} \cap \mathcal{O}^{-1}$; clearly $x \in F$ implies $F \subseteq E_x$. The second statement follows by Lemma 1. $\square$

From Lemma 2 and $(OC)_3$, the next lemma follows easily.

LEMMA 1.3. Assume that $\mathcal{F}, \mathcal{G}$ are filters on X such that: $\langle \mathcal{O} \rangle \subseteq \mathcal{F} \times \mathcal{F}$, $\langle \mathcal{O} \rangle \subseteq \mathcal{G} \times \mathcal{G}$, $E_x \in \mathcal{F}$, and $E_y \in \mathcal{G}$. The following statements are equivalent:

(1) $E_x = E_y$

(2) $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$

(3) $\langle \mathcal{O} \rangle \subseteq (\mathcal{F} \times \mathcal{G}) \cap (\mathcal{G} \times \mathcal{F})$.

THEOREM 1.4. $(X, \mathcal{O}, \mathcal{C})$ is a p.c.s. iff conditions (OC_1), $(OC)_2$, and $(OC)_3$ are satisfied.

PROOF. As we have noted previously, it is enough to prove that the three conditions are sufficient. From Proposition 2.6, (generalized to preordered sets), we see that $(X, \mathcal{O}, \mathcal{C})$ is a p.c.s. iff $\mathcal{O} = \cup\{\cap \mathcal{G} : \mathcal{G} \in \sigma_{\mathcal{O},\mathcal{C}}\}$ and $\mu_{\mathcal{O},\mathcal{C}}$ is compatible with $\mathcal{C}$, where $\sigma_{\mathcal{O},\mathcal{C}} = \sigma_{\mathcal{O}} \wedge \mu^{\mathcal{C}}$ is characterized in Proposition 2.4, [2] and $\mu_{\mathcal{O},\mathcal{C}} = (\sigma_{\mathcal{O},\mathcal{C}}) \vee (\sigma_{\mathcal{O},\mathcal{C}})^{-1}$. In view of Proposition 2.7, [2], it is sufficient to show that $\mu_{\mathcal{O},\mathcal{C}}$ is compatible with $\mathcal{C}$. In other words, we must show, as in the proof of Theorem 2.9, [2] that $\mathcal{H} \times \mathcal{H} \in \sigma_{\mathcal{O},\mathcal{C}}$ implies $\mathcal{H} \in \mathcal{C}$.

By Proposition 2.4, [2], we may assume $\mathcal{H} \times \mathcal{H} \supseteq \mathcal{G}$, where $\mathcal{G} = (\bigcap_{j=1}^{n} \mathcal{O}^{-1}(\mathcal{F}_j) \times \mathcal{O}(\mathcal{G}_j)) \cap \langle \mathcal{O} \rangle$, and $\mathcal{F}_j \lesssim \mathcal{G}_j$, $1 \le j \le n$. If $E_x \in \mathcal{H}$ for some $x \in X$, then $\mathcal{H} \in \mathcal{C}$ by Lemma 1. Suppose $E_x \notin \mathcal{H}$ for all $x \in X$. Then there must be an ultrafilter $\mathcal{K}$ finer than $\mathcal{H}$ such that $\langle \mathcal{O} \rangle \not\subseteq \mathcal{K} \times \mathcal{K}$. Otherwise, there must be (by Lemma 2) ultrafilters $\mathcal{K}_1$ and $\mathcal{K}_2$, both finer than $\mathcal{H}$, such that $\langle \mathcal{O} \rangle \subseteq \mathcal{K}_1 \times \mathcal{K}_1$, $\langle \mathcal{O} \rangle \subseteq \mathcal{K}_2 \times \mathcal{K}_2$, and, for nonequivalent elements x and y, $E_x \in \mathcal{K}_1$ and $E_y \in \mathcal{K}_2$. Since $\mathcal{K}_1 \times \mathcal{K}_2 \supseteq \mathcal{G}$ and $\mathcal{K}_2 \times \mathcal{K}_1 \supseteq \mathcal{G}$, one can show that $\mathcal{K}_1 \lesssim \mathcal{K}_2$ and $\mathcal{K}_2 \lesssim \mathcal{K}_1$. But then $(OC)_2$ requires that $\mathcal{K}_1 \cap \mathcal{K}_2 \in \mathcal{C}$, and it follows from Lemma 3 that $E_x \in \mathcal{K}_1 \cap \mathcal{K}_2$, a contradiction.

Let $\mathbb{K}$ be the set of all ultrafilters $\mathcal{K}$ finer than $\mathcal{H}$ such that $\langle \mathcal{O} \rangle \not\subseteq \mathcal{K} \times \mathcal{K}$. The argument given in paragraphs 2 and 3 of the proof of Theorem 2.9, [2] leads to the conclusion that there is $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M} \subseteq \mathcal{K}$ for all $\mathcal{K} \in \mathbb{K}$. Thus if $\mathcal{H}_1 = \cap\{\mathcal{K} : \mathcal{K} \in \mathbb{K}\}$, we conclude that $\mathcal{M} \subseteq \mathcal{H}_1$ and $\mathcal{H}_1 \in \mathcal{C}$.

If $\mathcal{H}_1 = \mathcal{H}$, the proof is complete. Assume instead that there is an ultrafilter $\mathcal{L}$ finer than $\mathcal{H}$ such that $\mathcal{L} \notin \mathcal{K}$. Then $\langle \mathcal{O} \rangle \subseteq \mathcal{L} \times \mathcal{L}$, and so $E_x \in \mathcal{L}$ for some

$x \in X$ by Lemma 2. A repetition of a previous argument shows that every such ultrafilter $\mathcal{L}$ must contain E_x for the same $x \in X$. If $\mathbb{L}$ is the set of all such ultrafilters $\mathcal{L}$ and $\mathcal{H}_2 = \cap\{\mathcal{L}: \mathcal{L} \in \mathbb{L}\}$, it follows that $E_x \in \mathcal{H}_2$, and so $\mathcal{H}_2 \in \mathcal{C}$.

It remains only to show that $\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2 \in \mathcal{C}$. Let $\mathcal{K} \in \mathbb{K}$ and $\mathcal{L} \in \mathbb{L}$. Then $\mathcal{K}$ and $\mathcal{L}$ are in $\mathcal{C}$, and $\mathcal{L} \times \mathcal{K} \supseteq \mathcal{G}$. As in an earlier paragraph of the proof, we can deduce that $\mathcal{K} \lesssim \mathcal{L}$ and $\mathcal{L} \lesssim \mathcal{K}$, from which it follows by $(OC)_2$ that $\mathcal{K} \cap \mathcal{L} \in \mathcal{C}$. Thus $\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2 \in \mathcal{C}$, and the proof is complete. $\square$

PROPOSITION 1.5. If $(X, \mathcal{O}, \mathcal{C})$ is a p.c.s., then $\mathcal{O}$ is closed in $X \times X$.

PROOF. It is shown in Proposition 2.7, [2], that $(OC)_3$ is equivalent to the statement : $\mathcal{O} = \cup\{\cap \mathcal{G}: \mathcal{G} \in \sigma_{\mathcal{O},\mathcal{C}}\}$. Since $\langle\mathcal{O}\rangle \in \sigma_{\mathcal{O},\mathcal{C}}$ and $\sigma_{\mathcal{O},\mathcal{C}}$ is closed under compositions $\mathcal{O} = \cup\{\cap \mathcal{B} \circ \langle\mathcal{O}\rangle \circ \mathcal{B}: \mathcal{B} \in \sigma_{\mathcal{O},\mathcal{C}}\} \supseteq \cup\{\cap \mathcal{D} \circ \langle\mathcal{O}\rangle \circ \mathcal{D}: \mathcal{D} \in \mu_{\mathcal{O},\mathcal{C}}\} = \overline{\mathcal{O}}$, where the last equality follows from 4.1.5, p. 301, [1], the closure being taken relative to the product convergence structure on $X \times X$ derived from the convergence structure on X compatible with $\mu_{\mathcal{O},\mathcal{C}}$. By Proposition 2.6, [1], the latter convergence structure is precisely $q_{\mathcal{C}}$. $\square$

Let (X,q) be a convergence space (in the sense of Fischer), let $\mathcal{C}^q$ be the set of all q-convergent filters, and let $q(x)$ be the set of all filters which q-converge to x. It is well known that the following statements are equivalent:

(a) There is a Cauchy structure $\mathcal{C}$ on X such that $q = q_{\mathcal{C}}$.

(b) $\mathcal{C}^q$ is a Cauchy structure on X.

(c) For $x, y \in X$, $q(x)$ and $q(y)$ are either equal or disjoint.

If $(X,\mathcal{O})$ is a preordered set and q a convergence structure on X, we define the triple $(X, \mathcal{O}, q)$ to be a <u>preordered convergence space</u> if there is a Cauchy structure $\mathcal{C}$ on X such that $(X,\mathcal{O},\mathcal{C})$ is a p.c.s. and $q = q_{\mathcal{C}}$.

PROPOSITION 1.6. Let $(X,\mathcal{O})$ be a preordered set, q a convergence structure on X. Then $(X, \mathcal{O}, q)$ is a preordered convergence space iff $(X, \mathcal{O}, \mathcal{C}^q)$ is a p.c.s.

PROOF. Let $(X, \mathcal{O}, q)$ be a preordered convergence space, and let $(X, \mathcal{O}, \mathcal{C})$ be a p.c.s. such that $q = q_{\mathcal{C}}$. Note that $\mathcal{C}^q \supseteq \mathcal{C}$. From the fact that $(X, \mathcal{O}, \mathcal{C})$ satisfies $(OC)_1$ and $(OC)_3$, it is easy to deduce that $(X, \mathcal{O}, \mathcal{C}^q)$ has the same properties. Furthermore the latter space also satisfies $(OC)_2$, since for complete spaces $(OC)_2$ follows immediately from $(OC)_3$.

The converse argument is trivial. $\square$

THEOREM 1.7. Let $(X, \mathcal{O})$ be a preordered set, q a convergence structure on X. Then $(X, \mathcal{O}, q)$ is a preordered convergence space iff $(X, \mathcal{O}, q)$ is locally convex and $\mathcal{O}$ is closed.

PROOF. Using Proposition 1.5, it is a simple matter to verify that the two properties specified for $(X, \mathcal{O}, q)$ are equivalent to the assertion that $(X, \mathcal{O}, \mathcal{C}^q)$ satisfies $(OC)_1$ and $(OC)_3$. Since $(X, \mathcal{O}, \mathcal{C}^q)$ is complete, Theorem 1.4 and Proposition 1.6 imply the desired conclusion. $\square$

2. WEAKLY PREORDERED CAUCHY SPACES (WEAK COMPATIBILITY).

As before, we assume that X is a set, $\mathcal{O}$ a preorder on X, and $\mathcal{C}$ a Cauchy structure on X. $(X, \mathcal{O}, \mathcal{C})$ is defined to be a <u>weakly preordered Cauchy space</u> if there is a quasi-uniform convergence structure σ on X such that: (1) the uniform

convergence structure $\mu = \sigma \vee \sigma^{-1}$ derived from σ has $\mathcal{C}$ as its set of Cauchy filters; (2) $\Theta = \cup\{\cap \mathcal{G} : \mathcal{G} \in \sigma\}$. This definition differs from that of preordered Cauchy space only in the condition $\langle\Theta\rangle \in \sigma$, which is required for the latter but not for our present definition. We abbreviate weakly preordered Cauchy space by "w.p.c.s.".

Given a preorder Θ on X, let τ_Θ be the quasi-uniform convergence structure on X with base consisting of all finite filter intersections of the form $\cap\{\dot{x}_i \times \dot{y}_i : (x_i, y_i) \in \Theta,\ i = 1,\ldots,n\}$. If, in addition, $\mathcal{C}$ is a Cauchy structure on X, let $\tau_{\Theta,\mathcal{C}} = \mu^{\mathcal{C}} \wedge \tau_\Theta$, where the lattice meet is taken in the lattice of all quasi-uniform convergence structures on X. Finally, we define a preorder on filters in $\mathcal{C}$ as follows:

$\mathcal{F} \textcircled{\leq} \mathcal{G}$ iff $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$, or there is $(x,y) \in \Theta$ such that

$$\mathcal{F} \cap \dot{x} \in \mathcal{C} \text{ and } \mathcal{G} \cap \dot{y} \in \mathcal{C}.$$

As we shall see, $\tau_{\Theta,\mathcal{C}}$ and $\textcircled{\leq}$ play the same role in the theory of weak compatibility that $\sigma_{\Theta,\mathcal{C}}$ and $\lesssim$, respectively, play in that of strong compatibility.

PROPOSITION 2.1. Let (X,Θ) be a preordered set with Cauchy structure $\mathcal{C}$. Then $\tau_{\Theta,\mathcal{C}}$ has a base of sets consisting of all finite intersections of the form $\cap\{\mathcal{F}_j \times \mathcal{G}_j : j = 1,\ldots,n\}$, where $\mathcal{F}_j \textcircled{\leq} \mathcal{G}_j$, $j = 1,\ldots,n$.

PROOF. Consider a composition of the form $\mathcal{G} = (\mathcal{G}_1 \circ \mathcal{G}_2 \circ \ldots \circ \mathcal{G}_n) \cap \dot{\Delta}$, where each $\mathcal{G}_i = \mathcal{F}_i \times \mathcal{G}_i$, and there are two possibilities for $\mathcal{F}_i$ and $\mathcal{G}_i$: (1) $\mathcal{F}_i = \dot{x}_i$ where $(x_i, y_i) \in \Theta$, or (2) $\mathcal{F}_i = \mathcal{G}_i$, and $\mathcal{F}_i \times \mathcal{F}_i \in \mu^{\mathcal{C}}$ (i.e., $\mathcal{F}_i \in \mathcal{C}$). By examining four possible cases, it is easy to verify that the existence of the composition $\mathcal{G}_i \circ \mathcal{G}_{i+1}$ implies that $\mathcal{F}_i \textcircled{\leq} \mathcal{G}_{i+1}$. Thus if $\mathcal{F} = \mathcal{F}_1$ and $\mathcal{G} = \mathcal{G}_n$, we obtain for the entire chain of compositions $\mathcal{F} \textcircled{\leq} \mathcal{G}$ and $\mathcal{F} \times \mathcal{G} = \mathcal{G}_1 \circ \mathcal{G}_2 \circ \ldots \circ \mathcal{G}_n$.

We thus see that filters of the form indicated in the proposition are in $\tau_{\Theta,\mathcal{C}}$, and a base for $\tau_{\Theta,\mathcal{C}}$ involves taking finite compositions and intersections of such filters. It can be shown by a straightforward set theoretic argument that any finite composition of filters of the indicated form can again be written as a finite intersection of filters of the same form, which is the desired conclusion. $\square$

PROPOSITION 2.2. A triple $(X, \Theta, \mathcal{C})$ is a w.p.c.s. iff $\Theta = \cup\{\cap \mathcal{F} : \mathcal{F} \in \tau_{\Theta,\mathcal{C}}\}$, and $\mathcal{C}$ is compatible with $\nu_{\Theta,\mathcal{C}} = \tau_{\Theta,\mathcal{C}} \vee (\tau_{\Theta,\mathcal{C}})^{-1}$.

PROOF. Given the two conditions, $(X, \Theta, \mathcal{C})$ is a w.p.c.s. according to the definition of this term. Conversely, assume that $(X, \Theta, \mathcal{C})$ is a w.p.c.s. derived from a quasi-uniform convergence structure σ, and let $\mu = \sigma \vee \sigma^{-1}$ be the associated uniform convergence structure. Then it is easy to see that $\mu \leq \nu_{\Theta,\mathcal{C}} \leq \mu^{\mathcal{C}}$, and since μ and $\mu^{\mathcal{C}}$ are compatible with $\mathcal{C}$, so is $\nu_{\Theta,\mathcal{C}}$. Also $\Theta = \cup\{\cap \mathcal{G} : \mathcal{G} \in \sigma\} \supseteq \cup\{\cap \mathcal{G} : \mathcal{G} \in \tau_\Theta \wedge \mu\} \supseteq \cup\{\cap \mathcal{G} : \mathcal{G} \in \tau_{\Theta,\mathcal{C}}\} \supseteq \Theta$, since $(x,y) \in \Theta \Rightarrow \dot{x} \times \dot{y} \in \tau_\Theta \subseteq \tau_{\Theta,\mathcal{C}}$. $\square$

We next introduce conditions on a triple $(X, \Theta, \mathcal{C})$ which lead to a characterization of a w.p.c.s. similar to that given for a p.c.s. in Theorem 1.4. It turns out that only two such conditions are needed $(\text{woc})_2$ and $(\text{woc})_3$ which are analogous to $(\text{OC})_2$ and $(\text{OC})_3$, respectively. There is no form of "local convexity" involved in the characterization of weak compatibility, and so we shall later use $(\text{woc})_1$ to describe a single condition (not related to $(\text{OC})_1$) which can replace both $(\text{woc})_2$ and $(\text{woc})_3$.

$(\mathrm{woc})_2$ If $\mathcal{F}, \mathcal{G}$ are in $\mathcal{C}$, $\mathcal{F} \circledleq \mathcal{G}$, and $\mathcal{G} \circledleq \mathcal{F}$, then $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$.

$(\mathrm{woc})_3$ $\dot{x} \circledleq \dot{y}$ implies $(x,y) \in \mathcal{O}$.

PROPOSITION 2.3. If $(X, \mathcal{O}, \mathcal{C})$ is a w.p.c.s., then $(\mathrm{woc})_2$ and $(\mathrm{woc})_3$ are satisfied.

PROOF. Assume $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, $\mathcal{F} \circledleq \mathcal{G}$, and $\mathcal{G} \circledleq \mathcal{F}$. Then, by Proposition 2.1, $\mathcal{F} \times \mathcal{G} \in \tau_{\mathcal{O},\mathcal{C}}$ and $\mathcal{G} \times \mathcal{F} \in \tau_{\mathcal{O},\mathcal{C}}$. Thus $(\mathcal{F} \cap \mathcal{G}) \times (\mathcal{F} \cap \mathcal{G}) = (\mathcal{F} \times \mathcal{G}) \cap (\mathcal{G} \times \mathcal{F}) \cap (\mathcal{F} \times \mathcal{F}) \cap (\mathcal{G} \times \mathcal{G}) \in \tau_{\mathcal{O},\mathcal{C}}$, and by Proposition 2.2, $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$. Next, let $\dot{x} \circledleq \dot{y}$. By Proposition 2.1, $\dot{x} \times \dot{y} \in \tau_{\mathcal{O},\mathcal{C}}$, and so $(x,y) \in \mathcal{O}$. $\square$

THEOREM 2.4. $(X, \mathcal{O}, \mathcal{C})$ is a w.p.c.s. iff conditions $(\mathrm{woc})_2$ and $(\mathrm{woc})_3$ are satisfied.

PROOF. Using Proposition 2.2, we must show:

(1) $\mathcal{O} = \bigcup\{\bigcap \mathcal{G} : \mathcal{G} \in \tau_{\mathcal{O},\mathcal{C}}\}$;

(2) $\mathcal{C} = \mathcal{C}_\mu$, where $\mu = \nu_{\mathcal{O},\mathcal{C}} = \tau_{\mathcal{O},\mathcal{C}} \vee (\tau_{\mathcal{O},\mathcal{C}})^{-1}$.

(1). If $(x,y) \in \mathcal{O}$, then $\dot{x} \times \dot{y} \in \tau_{\mathcal{O},\mathcal{C}}$, which implies $(x,y) \in \bigcup\{\bigcap \mathcal{G} : \in \tau_{\mathcal{O},\mathcal{C}}\}$. If $(x,y) \in \bigcap \mathcal{G}$ for some $\mathcal{G} \in \tau_{\mathcal{O},\mathcal{C}}$, then $\dot{x} \times \dot{y} \supseteq \mathcal{G}$, implying $\dot{x} \times \dot{y} \in \tau_{\mathcal{O},\mathcal{C}}$. By Proposition 2.1, $\dot{x} \circledleq \dot{y}$, and so $(x,y) \in \mathcal{O}$ by $(\mathrm{woc})_3$.

(2) It is sufficient to show $\mathcal{H} \times \mathcal{H} \in \tau_{\mathcal{O},\mathcal{C}}$ implies $\mathcal{H} \in \mathcal{C}$. By Proposition 2.1, we may assume $\mathcal{H} \times \mathcal{H} \supseteq \mathcal{G} = (\bigcap_{j=1}^{n} (\mathcal{F}_j \times \mathcal{G}_j)) \cap \dot{\Delta}$, where $\mathcal{F}_j \circledleq \mathcal{G}_j$, for $j = 1,\ldots,n$. Let $\mathcal{K}$, $\mathcal{L}$ be distinct ultrafilters finer than $\mathcal{H}$. Then there are indices i, j such that $\mathcal{F}_i \times \mathcal{G}_i \subseteq \mathcal{F} \times \mathcal{L}$ and $\mathcal{F}_j \times \mathcal{G}_j \subseteq \mathcal{L} \times \mathcal{F}$, where $\mathcal{F}_i \circledleq \mathcal{G}_i$ and $\mathcal{F}_j \circledleq \mathcal{G}_j$. This implies $\mathcal{K} \circledleq \mathcal{L}$ and $\mathcal{L} \circledleq \mathcal{K}$, so that $\mathcal{K} \cap \mathcal{L} \in \mathcal{C}$ by $(\mathrm{woc})_2$. Thus $\mathcal{F}_i \cap \mathcal{G}_i \cap \mathcal{F}_j \cap \mathcal{G}_j \in \mathcal{C}$.

Rearranging the indices if necessary, let $1,\ldots,m$ be the indices such that, if $j \le m$, then there are ultrafilters $\mathcal{K}, \mathcal{L}$ finer than $\mathcal{H}$ such that $\mathcal{K} \times \mathcal{L} \supseteq \mathcal{F}_j \times \mathcal{G}_j$, where $\mathcal{F}_j \circledleq \mathcal{G}_j$ in $\mathcal{C}$. Let $\mathcal{M}_j = \mathcal{F}_j \cap \mathcal{G}_j$ and $\mathcal{M} = \bigcap_{j=1}^{m} \mathcal{M}_j$. The reasoning of the preceding paragraph leads to the conclusion that $\mathcal{M} \in \mathcal{C}$ and that $\mathcal{K} \supseteq \mathcal{M}$ for every ultrafilter $\mathcal{K} \supseteq \mathcal{H}$. Thus $\mathcal{H} \supseteq \mathcal{M}$, which implies $\mathcal{H} \in \mathcal{C}$, and the proof is complete. $\square$

We now introduce the condition $(\mathrm{woc})_1$ for a triple $(X, \mathcal{O}, \mathcal{C})$.

$(\mathrm{woc})_1$ $q_{\mathcal{C}}(x) = q_{\mathcal{C}}(y)$ iff $(x,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$.

It is significant that this condition is formulated entirely in terms of $\mathcal{O}$ and $q_{\mathcal{C}}$, the convergence structure derived from $\mathcal{C}$.

THEOREM 2.5. $(X, \mathcal{O}, \mathcal{C})$ is a w.p.c.s. iff condition $(\mathrm{woc})_1$ is satisfied.

PROOF. In both directions of this proof, we use the characterization of a w.p.c.s. given in Theorem 2.4.

Assume that $(X, \mathcal{O}, \mathcal{C})$ satisfies $(\mathrm{woc})_2$ and $(\mathrm{woc})_3$. If $q_{\mathcal{C}}(x) = q_{\mathcal{C}}(y)$, then $\dot{x} \to y$ in $q_{\mathcal{C}}$. Thus $\dot{x} \cap \dot{y} \in \mathcal{C}$, which implies $\dot{x} \le \dot{y}$ and $\dot{y} \le \dot{x}$. By $(\mathrm{woc})_3$, $(x,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$. If $(x,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$ and $\mathcal{F} \in q_{\mathcal{C}}(x)$, then $\dot{y} \circledleq \mathcal{F} \cap \dot{x}$ and $\mathcal{F} \cap \dot{x} \circledleq \dot{y}$. By $(\mathrm{woc})_2$, $\dot{y} \cap \dot{x} \cap \mathcal{F} \in \mathcal{C}$, implying $\mathcal{F} \in q_{\mathcal{C}}(y)$. By the symmetric argument, $\mathcal{G} \in q_{\mathcal{C}}(y)$ implies $\mathcal{G} \in q_{\mathcal{C}}(x)$, and so $q_{\mathcal{C}}(x) = q_{\mathcal{C}}(y)$. Thus $(\mathrm{woc})_1$ is established.

Conversely, assume that $(X,\mathcal{O},\mathcal{C})$ satisfies $(\text{woc})_1$. Let $\mathcal{F} \circledleq \mathcal{G}$ and $\mathcal{G} \circledleq \mathcal{F}$, where $\mathcal{F}, \mathcal{G} \in \mathcal{C}$. Considering only the nonobvious case, we can assume that there is $(x,y) \in \mathcal{O}$ such that $\mathcal{F} \cap \dot{x} \in \mathcal{C}$ and $\mathcal{G} \cap \dot{y} \in \mathcal{C}$. We can also assume there is $(b, a) \in \mathcal{O}$ such that $\mathcal{F} \cap \dot{a} \in \mathcal{C}$ and $\mathcal{G} \cap \dot{b} \in \mathcal{C}$. Then $\mathcal{F} \in q_{\mathcal{C}}(a) \cap q_{\mathcal{C}}(x) \Rightarrow q_{\mathcal{C}}(a) = q_{\mathcal{C}}(x)$. Likewise, $q_{\mathcal{C}}(b) = q_{\mathcal{C}}(y)$. By $(\text{woc})_1$, $(a,x) \in \mathcal{O} \cap \mathcal{O}^{-1}$ and $(b,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$. Thus (x,a), (a,b), (b,y) are all in $\mathcal{O}^{-1}$, which implies $(x,y) \in \mathcal{O}^{-1}$. Thus $(x,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$, and again by $(\text{woc})_1$ we conclude $q_{\mathcal{C}}(x) = q_{\mathcal{C}}(y)$. Consequently, $\mathcal{F} \cap \mathcal{G} \in q_{\mathcal{C}}(x) = q_{\mathcal{C}}(y)$, implying $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$, and so $(\text{woc})_2$ is established.

To prove $(\text{woc})_3$, let $\dot{x} \circledleq \dot{y}$. If $\dot{x} \cap \dot{y} \in \mathcal{C}$, then $q(x) = q(y)$, and $(x,y) \in \mathcal{O}$ follows by $(\text{woc})_1$. Otherwise there are elements a, b in X such that $(a,b) \in \mathcal{O}$ and $\dot{x} \cap \dot{a} \in \mathcal{C}$, $\dot{y} \cap \dot{b} \in \mathcal{C}$. But this implies $q(a) = q(x)$, so that $(x,a) \in \mathcal{O} \cap \mathcal{O}^{-1}$, and $q_{\mathcal{C}}(y) = q_{\mathcal{C}}(b)$, so that $(y,b) \in \mathcal{O} \cap \mathcal{O}^{-1}$. Thus (x,a), (a,b), (b,y) are all in $\mathcal{O}$, and so $(x,y) \in \mathcal{O}$. $\square$

Next, we consider weak compatibility for convergence spaces. Let $(X,\mathcal{O})$ be a preordered set, and let q be a convergence structure on X. The triple $(X, \mathcal{O}, q)$ is a <u>weakly preordered convergence space</u> if there is a Cauchy structure $\mathcal{C}$ on X such that $(X, \mathcal{O}, \mathcal{C})$ is a w.p.c.s. and $q = q_{\mathcal{C}}$.

If $(X, \mathcal{O}, q)$ is a weakly preordered convergence space, then it is an immediate consequence of Theorem 2.5 that every Cauchy structure $\mathcal{C}$ compatible with q has the property that $(X, \mathcal{O}, \mathcal{C})$ is a w.p.c.s.; in particular $(X, \mathcal{O}, \mathcal{C}^q)$ has this property. Thus we obtain

COROLLARY 2.6. For a preordered set $(X,\mathcal{O})$ with convergence structure q, the following statements are equivalent.

(a) $(X, \mathcal{O}, q)$ is a weakly preordered convergence space.

(b) $(X, \mathcal{O}, \mathcal{C}^q)$ is a w.p.c.s.

(c) If $\mathcal{C}$ is a Cauchy structure on X such that $q = q_{\mathcal{C}}$, then $(X,\mathcal{O},\mathcal{C})$ is a w.p.c.s.

One final characterization of weakly preordered convergence space which makes no reference to the notion of Cauchy structure is given in the next theorem.

THEOREM 2.7. The triple $(X,\mathcal{O}, q)$ is a weakly preordered convergence space iff the following conditions are satisfied:

(a) $(x,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$ implies $q(x) = q(y)$;

(b) $(x,y) \notin \mathcal{O} \cap \mathcal{O}^{-1}$ implies $q(x) \cap q(y) = \phi$.

PROOF. If $(X, \mathcal{O}, q)$ is a weakly preordered convergence space, then (a) follows immediately from Corollary 2.6, and (b) follows from Corollary 2.6 and the fact that $q = q_{\mathcal{C}}$ for some Cauchy structure $\mathcal{C}$.

Conversely, assume the two conditions; it suffices to show that $(X, \mathcal{O}, \mathcal{C}^q)$ satisfies condition $(\text{woc})_1$. From (a) and (b), we see that for any x, y $\in$ X, q(x) and q(y) are either equal or disjoint; thus $\mathcal{C}^q$ is a Cauchy structure. Also (a) gives "half" of condition $(\text{woc})_1$, and the other "half" comes from (b), since $q(x) = q(y)$ implies $q(x) \cap q(y) \neq \phi$, and so $(x,y) \in \mathcal{O} \cap \mathcal{O}^{-1}$. $\square$

In case $\mathcal{O}$ is a partial order, it is appropriate to speak of a <u>weakly ordered Cauchy space</u> or a <u>weakly ordered convergence space</u> in the case of weak order compatibility. In this case the compatibility is indeed "weak", since any triple $(X, \mathcal{O}, \mathcal{C})$ where $(X, \mathcal{O})$ is any poset and $\mathcal{C}$ any T_1 Cauchy structure will constitute a weakly ordered Cauchy space, and likewise if q is any T_2 convergence space then

$(X, \mathcal{O}, \mathcal{C})$ will be a weakly ordered convergence space. It is well known that, in general, every T_1 Cauchy space has a variety of T_1 completions. Thus if $(X, \mathcal{O}, \mathcal{C})$ is any weakly ordered Cauchy space and $(X', \mathcal{C}')$ is any T_1 completion of $(X, \mathcal{C})$, then a partial order $\mathcal{O}'$ can be defined on X' in many ways so as to make $(X', \mathcal{O}', \mathcal{C}')$ a weakly ordered Cauchy completion of $(X, \mathcal{O}, \mathcal{C})$.

In the case of strong compatibility the completion theory is much more complicated. Not all ordered Cauchy spaces have ordered Cauchy completions. A detailed treatment of this topic may be found in [2].

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