ON M-IDEALS IN $B(\sum_{i=1}^{\infty} \oplus_p \ell_r^{n_i})$

CHONG-MAN CHO

Department of Mathematics
College of Natural Science
Hangyang University
Seoul 133, Korea

(Received May 8, 1986)

ABSTRACT. For $1 < p, r < \infty$, $X = (\sum_{i=1}^{\infty} \oplus_p \ell_r^{n_i})$, $\{n_i\}$ bounded, the space $K(X)$ of all compact operators on X is the only nontrivial M -ideal in the space $B(X)$ of all bounded linear operators on X .

KEY WORDS AND PHRASES. Compact operators, hermitian element, M -ideal.

1980 AMS SUBJECT CLASSIFICATION CODE. Primary 46A32, 47B05, secondary 47B05.

1. INTRODUCTION.

Since Alfsen and Effros [1] introduced the notion of an M -ideal, many authors have studied M -ideals in operator algebras. It is known that $K(X)$, the space of all compact operators on X , is an M -ideal in $B(X)$, the space of all bounded linear operators on X , if X is a Hilbert space or ℓ_p ($1 < p < \infty$). Smith and Ward [2] proved that M -ideals in a C^* -algebra are exactly the closed two sided ideals. Smith and Ward [3], and Flinn [4] proved that, for $1 < p < \infty$, $K(\ell_p)$ is the only nontrivial M -ideal in $B(\ell_p)$. The purpose of this paper is to generalize this result to $B(X)$, where $X = (\sum_{i=1}^{\infty} \oplus_p \ell_r^{n_i})$, for $1 < p, r < \infty$ and $\{n_i\}$ a bounded sequence of positive integers. In this proof, the ideas and results of [4], [2], [5] and [3] are heavily used.

2. NOTATIONS AND PRELIMINARIES.

If X is a Banach space, $B(X)$ (resp. $K(X)$) will denote the space of all bounded linear operators (resp. compact linear operators) on X .

A closed subspace J of a Banach space X is an L -summand (resp. M -summand) if there is a closed subspace $\tilde{J}$ of X such that X is the algebraic direct sum of J and $\tilde{J}$, and $\|x + y\| = \|x\| + \|y\|$ (resp. $\|x\| = \max\{\|x\|, \|y\|\}$) for $x \in J, y \in \tilde{J}$. A projection $P: X \rightarrow X$ is an L -projection (resp. M -projection) if $\|x\| = \|Px\| + \|(I - P)x\|$ (resp. $\|x\| = \max\{\|Px\|, \|(I - P)x\|\}$ for every $x \in X$).

A closed subspace J of a Banach space X is an M -ideal in X if $J^\perp = \{x^* \in X^*: x^*|_J = 0\}$ is an L -summand in X^* .

If $(X_i)_{i=1}^\infty$ is a sequence of Banach spaces for $1 \leq p \leq \infty$, $\sum_{i=1}^\infty \oplus_p X_i$ is the space of all sequences $x = (x_i)_{i=1}^\infty$, $x_i \in X_i$, with the norm $\|x\| = (\sum_{i=1}^\infty \|x_i\|^p)^{1/p} < \infty$ if $1 \leq p < \infty$ and $\|x\| = \sup_i \|x_i\| < \infty$ if $p = \infty$.

An element h in a complex Banach algebra A with the identity e is hermitian if $\|e^{i\lambda h}\| = 1$ for all real λ [6].

If J_1 and J_2 are complementary nontrivial M -summands in A (i.e. $A = J_1 \oplus J_2$), P is the M -projection of A onto J_1 and $z = P(e) \in J_1$, then z is hermitian with $z = z^2$ [2, 3.1], $zJ_i \subseteq J_i$ ($i = 1, 2$) and $zJ_2z = 0$ [2, 3.2 and 3.4]. Since $I - P$ is the M -projection of A onto J_2 , $e - z = (e - z)^2$ is hermitian, $(e - z)J_i \subseteq J_i$ ($i = 1, 2$) and

$$(e - z)J_1(e - z) = 0.$$

If M is an M -ideal in a Banach algebra A , then M is a subalgebra of A [2, 3.6]. If $h \in A$ is hermitian and $h^2 = e$, then $hM \subseteq M$ and $Mh \subseteq M$ [4, Lemma 1].

If A is a Banach algebra with the identity e , then A^{**} endowed with Arens multiplication is a Banach algebra and the natural embedding of A into A^{**} is an algebra isomorphism into [6]. If J is an M -ideal in A , then $A^{**} = J^{\perp\perp} \oplus_\infty (J^{\perp\perp})^\sim$ and the associated hermitian element $z \in J^{\perp\perp}$ commutes with every other hermitian element of A^{**} [5, 22].

From now X , will always denote $\sum_{i=1}^\infty \oplus_p \ell_r^{n_i}$, where $1 < p, r < \infty$ and $\{n_i\}_{i=1}^\infty$ a bounded sequence of positive integers. An operator $T \in B(X)$ has a matrix representation with respect to the natural basis of X . From the definition, it is obvious that any diagonal matrix $T \in B(X)$ with real entries is hermitian.

Flinn [4] proved that if M is an M -ideal in $B(\ell_p)$ and $h \in B(\ell_p)$ is a diagonal matrix, then $hM \subseteq M$ and $Mh \subseteq M$. His proof is valid for X . He also proved that if M is a nontrivial M -ideal in $B(\ell_p)$, then $M \supseteq K(\ell_p)$. Again his proof with a small modification is valid for X .

Thus we have observed that if M is a nontrivial M -ideal in $B(X)$, then $M \supseteq K(X)$.

If M is an M -ideal in a Banach algebra A and $h \in M$ is hermitian, then $hAh \subseteq M$.

Indeed, $(e - z)h = (e - z)^2h = (e - z)h(e - z) = 0 = h(e - z)$ and so $zh = hz = h$.

Since $zA^{**}z \subseteq M^{\perp\perp}$ [2: 3.4], $zAz \subseteq M^{\perp\perp}$ and hence $hAh = hzAz = M^{\perp\perp}$. Since $h \in M$,

$hAh \subseteq A \cap M^{\perp\perp} = M$. Thus if $e \in M$, then $A = M$.

3. MAIN THEOREM.

We may assume that $X = (\ell_r^{m_1} \oplus \dots \oplus \ell_r^{m_s}) \oplus (\ell_r^{n_1} \oplus \dots \oplus \ell_r^{n_k}) \oplus (\ell_r^{n_1} \oplus \dots \oplus \ell_r^{n_k}) \oplus \dots$

Set $\alpha = m_1 + \dots + m_s$ and $\beta = n_1 + \dots + n_k$. Let N be the set of all natural numbers, $S_0 = \{1, 2, \dots, \alpha\}$ and, for $1 \leq j \leq k$, $S_j = \bigcup_n (n + \beta N)$, where n runs over $\alpha + n_0 + n_{j-1} < n \leq \alpha + n_0 + \dots + n_j$, $n_0 = 0$. Let P_j be the projection on X defined by $P_j x = 1_{S_j} x$ for every $x \in X$, where 1_{S_j} is the indicator function of the set S_j . Let $(e_i)_{i=1}^\infty$ be the unit vector basis for X . $A = \sum_{ij} a_{ij} e_j \otimes e_i \in B(X)$ is the operator with matrix (a_{ij}) with respect to $(e_i)_{i=1}^\infty$.

LEMMA 1. If M is an M -ideal in $B(X)$ and contains $A = \sum a_{ij} e_j \otimes e_i$ such that $(a_{ii})_{i \geq 1} \in \ell_\infty \setminus c_0$, then $M = B(X)$.

PROOF. By multiplying by diagonal matrices from both sides, and as in Lemma 2 [4], we may assume that $A = \sum_{i=1}^\infty e_{f(i)} \otimes e_{f(i)}$, where $f(i+1) - f(i) \geq \beta$, $f(i) \in S_j$ for all i and a fixed j ($1 \leq j \leq k$). Fix ℓ ($\ell \neq j$, $1 \leq \ell \leq k$) and s

$(\alpha + n_0 + \dots + n_{\ell-1} < s \leq \alpha + n_0 + \dots + n_\ell)$, and let $g(i) = s + (i-1)\beta$ ($i = 1, 2, 3, \dots$).

CLAIM: $B = \sum_{i=1}^\infty e_{g(i)} \otimes e_{f(i)} \in M$. Suppose $B \notin M$. Choose $\Phi \in M^\perp$ so that $\|\Phi\| = 1 = \Phi(B)$. Since $\|B\| = 1$ and $AB = B$, $\Psi \in B(X)^*$ defined by $\Psi(G) = \Phi(GB)$ has norm one and attains its norm at $A \in M$. Hence $\Psi \in \tilde{M}$ and $\|\Phi + \Psi\| = 2$, where

$B(X)^* = M^\perp \oplus_1 \tilde{M}$. Since $|(\Phi + \Psi)(G)| = |\Phi(G + GB)| \leq \|\Phi\| \|G\| \|I + B\|$,

$\|\Phi + \Psi\| \leq \|I + B\|$. To draw a contradiction, we will show that $\|I + B\| < 2$. Let j and ℓ be as above. For $x \in X$ with $\|x\| = 1$, $\|x\|^p = \|\dot{P}_j x\|^p + \|(I - P_j)x\|^p$. Let $t = \|P_j x\|^p$, then $1 - t = \|(I - P_j)x\|^p$. Since Bx has support in S_j and

$\|Bx\| \leq \|(I - P_j)x\|$, we have

$$\|(I + B)x\| \leq 1 + \|Bx\| \leq 1 + (1 - t)^{1/p} \quad (3.1)$$

$\|(I - P_j)x + Bx\| \leq (2\|(I - P_j)x\|)^{1/p} = 2^{1/p}(1 - t)^{1/p}$. Hence

$$\|(I + B)x\| = \|x + Bx\| \leq \|P_j x\| + \|(I - P_j)x + Bx\| \leq t^{1/p} + 2^{1/p}(1 - t)^{1/p} \quad (3.2)$$

Obviously, $F(t) = t^{1/p} + 2^{1/p}(1 - t)^{1/p}$ is continuous on $[0, 1]$ and $F(0) = 2^{1/p} < 2$ so $F(t) < 2$ for all $0 \leq t \leq \delta$. For $\delta \leq t \leq 1$, $1 + (1 - t)^{1/p} < 2$. By (3.1) and (3.2) above, $\|(I + B)\| < 2$. Contradiction! Hence $B \in M$.

Similarly $C = \sum_{i=1}^\infty e_{f(i)} \otimes e_{g(i)} \in M$ (use $\|C\| = 1$, $CA = C$, $\Psi(G) = \Phi(CG)$, $I + C$ is the adjoint of $I + B$). Hence $\|I + C\| < 2$.

Since M is an algebra, $1_{s+\beta N} \cdot I = CB \in M$. Thus for all $i = \alpha+1, \alpha+2, \dots, \alpha+\beta$, $1_{i+\beta N} \cdot I \in M$. Since $1_{S_0} \cdot I$ is compact, $1_{S_0} \cdot I \in M$. This proves $M = B(X)$.

COROLLARY 2. If M is an M -ideal in $B(X)$ and there exists an isometry

$\tau: B(X) \rightarrow B(X)$ so that $\tau(M)$ contains an $A = \sum_{i,j} a_{ij} e_j \otimes e_i$ with $(a_{ii})_{i \geq 1} \in \ell_\infty \setminus c_0$,

then $M = B(X)$.

PROOF. Since $\tau(M)$ is an M -ideal in $B(X)$ and $A \in \tau(M)$, by the lemma $\tau(M) = B(X)$.

Hence $M = B(X)$.

THEOREM 3. If M is an M -ideal in $B(X)$ and contains a noncompact $T = \sum t_{ij} e_j \otimes e_i$, then $M = B(X)$.

PROOF. Suppose $T \in M$ and T is not compact. Wlog we may assume

$$T = \sum_{k=1}^{\infty} T_k, \quad T_k = \sum_{j=m_k+1}^{m_k+n_k} t_{ij} e_j \otimes e_i, \quad \|T_k\| = 1 \text{ where } m_k \in \alpha + \beta\mathbb{N}, \quad n_k \in \beta\mathbb{N}, \text{ and} \\ m_k + n_k + \beta < m_{k+1}.$$

Since each T_k has norm one, there exists norm one vectors

$$x_k = (x_1^k) \in X, \quad y_k = (y_1^k) \in X^*, \quad z_k = (z_1^k) \in X^* \text{ all with supports in } \sigma_k = \{i: m_k < i \leq m_k + n_k\}$$

so that $y_k(T_k x_k) = 1 = z_k(x_k)$.

$$\text{Let } B_k = \sum_{j \geq 1} x_j^k e_{m_k+1} \otimes e_j, \quad C_k = \sum_{j \geq 1} y_j^k e_j \otimes e_{m_k+1}, \quad D_k = \sum_{j \geq 1} z_j^k e_j \otimes e_{m_k+1},$$

$$A = \sum_{k \geq 1} e_{m_k+1} \otimes e_{m_k+1}, \quad B = \sum_{k \geq 1} B_k, \quad C = \sum_{k \geq 1} C_k \text{ and } D = \sum_{k \geq 1} D_k. \text{ Then all of these operators}$$

have norm one and $DB = CTB = A$

Let P be the matrix obtained from the identity matrix I by interchanging (m_k+j) -th column and $(m_k + n_k + j)$ -th column for all k and $j(1 \leq j \leq \beta)$. Then P is an isometry in X since $n_k \in \beta\mathbb{N}$.

CLAIM. If $B \in M$, then $M = B(X)$.

Choose $\phi \in c_0^\perp \subseteq \ell_\infty^*$ so that $\|\phi\| = 1 = \phi((1,1,1,1,\dots))$. Define norm one functional

$\gamma \in B(X)^*$ by $\gamma(G) = \phi((g_{m_k+n_k+1, m_k+1})_{k \geq 1})$ where $G = \sum g_{ij} e_j \otimes e_i$. Then $\gamma \notin M^\perp$. In

fact, if $\gamma \in M^\perp$, then $\gamma_1 \in B(X)^*$ defined by $\gamma_1(G) = \phi((DG)_{m_k+1, m_k+1})$ has norm one and

attains its norm at $B \in M$. Hence $\gamma_1 \in \tilde{M}$ and $\|\gamma + \gamma_1\| = 2$. But for any norm one

$G \in B(X)$, we have

$$\begin{aligned} |(\gamma + \gamma_1)(G)| &= |\phi(g_{m_k+n_k+1, m_k+1} + \sum_{j \in \sigma_k} z_j^k g_{j, m_k+1})_{k \geq 1}| \\ &\leq \sup_k \|z_k + e_{m_k+n_k+1}\| (z_k + e_{m_k+n_k+1} \in X^*, \|G\| = 1) \\ &= 2^{1/p'} \text{ where } \frac{1}{p} + \frac{1}{p'} = 1. \end{aligned}$$

so $\|\gamma + \gamma_1\| \leq 2^{1/p'}$ contradiction! Thus $\gamma \notin M^\perp$. Since $\gamma \notin M^\perp$, there is $G \in M$ s.t. $\gamma(G) \neq 0$. So $(g_{m_k+n_k+1, m_k+1})_{k \geq 1} \in \ell_\infty \setminus c_0$. The sequence of the diagonal entries of $P(G)$ belongs to $\ell_\infty \setminus c_0$. Thus by corollary 2, $M = B(X)$. This proves the claim.

Next $\Psi \in B(X)^*$ defined by $\Psi(G) = \phi(((CG)_{m_k+1, m_k+n_k+1})_{k \geq 1})$ is not in $M^\perp$. Indeed, if $\Psi \in M^\perp$, then since $\Psi_1 \in B(X)^*$ defined by $\Psi_1(G) = \phi(((CGB)_{m_k+1, m_k+1})_{k \geq 1})$ has norm one and attains its norm at $T \in M$, $\Psi_1 \in \tilde{M}$ and so $\|\Psi + \Psi_1\| = 2$. But for any norm one $G \in B(X)$, we have

$$\begin{aligned} |(\Psi + \Psi_1)(G)| &\leq \sup_k |(CG)_{m_k+1, m_k+n_k+1} + \sum_{j \in \sigma_k} (CG)_{m_k+1, j} x_j^k| \\ &\leq \sup_k \|x^k + e_{m_k+n_k+1}\| \quad \text{since } CG \in B(X), \|CG\| = 1 \\ &= 2^{1/p}, \text{ contradiction!} \end{aligned}$$

Thus $\Psi \notin M^\perp$. So there is $G = \sum g_{ij} e_j \otimes e_i \in M$ such that $((CG)_{m_k+1, m_k+n_k+1})_{k \geq 1} \in \ell_\infty \setminus c_0$.

There is $\varepsilon > 0$ such that $\|G_k\| > \varepsilon$ for infinitely many k , where

$G_k = \sum_{j \in \sigma_k} g_j, m_k+n_k+1 e_{m_k+n_k+1} e_j$. We can choose diagonal matrices D_1 and D_2 in $B(X)$ so that $D_1 G D_2$ has the same form as B in the claim above. Since $D_1 G D_2 \in M$, $M = B(X)$.

REFERENCES

1. ALFSEN, E. and EFFROS, E. Structure in real Banach space, Ann. of Math. 96(1972), 98-173.
2. SMITH, R. and WARD, J. M-ideal structure in Banach algebras, J Functional Analysis 27(1978), 337-349.
3. SMITH, R. and WARD, J. Applications of convexity and M-ideal theory to quotient Banach algebras, Quart. J. Math. 30(1979), 365-384.
4. FLINN, P. A characterization of M-ideals in $B(\ell_p)$ for $1 < p < \infty$, Pacific J. Math. 98(1982), 73-80.
5. SMITH, R. and WARD, J. M-ideas in $B(\ell_p)$, Pacific J. Math 81(1979), 227-237.
6. BONSALL and DUNCAN Numerical Range of Operators on Normed space, London Math.Soc. Lecture Note Series 2, Cambridge (1971).
7. BEHREND, E. M-structure and the Banach-stone Theorem, Lecture notes in Mathematics 736, Springer-Verlag (1979).
8. BONSALL and DUNCAN Complete Normed Algebra, Ergebnisse der Math., 80, Springer-Verlag (1973).
9. CHO, Chong-Man and JOHNSON, W.B. A characterization of subspace of ℓ_p for which $K(X)$ is an M-ideal is $L(X)$, Proc. Amer. Math. Soc. Vol. 93(1985), 466-470.
10. LIMA, A. Intersection properties of balls and subspaces of Banach spaces, Trans. Amer. Math. Soc. 227(1977), 1-62.
11. LIMA, A. M-ideals of compact operators in classical Banach spaces, Math. Scand. 44(1979), 207-217.