GENERALIZATION OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

TADAYUKI SEKINE
Department of Mathematics
Science and Technology
Nihon University
1-8 Kanda Surugadai, Chiyoda-ku
Tokyo 101, Japan

(Received June 16, 1986 and in revised form September 18, 1986)

ABSTRACT. We introduce the subclass $T_j(n,m,a)$ of analytic functions with negative coefficients by the operator $D^n$. Coefficient inequalities and distortion theorems of functions in $T_j(n,m,a)$ are determined. Further, distortion theorems for fractional calculus of functions in $T_j(n,m,a)$ are obtained.

KEYWORDS AND PHRASES. Analytic functions, negative coefficients, coefficient inequalities, distortion theorem, fractional calculus.


1. INTRODUCTION.

Let $A_j$ denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N} = \{1, 2, 3, \ldots\}) \quad (1.1)$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

For a function $f(z)$ in $A_j$, we define

$$D^0 f(z) = f(z), \quad (1.2)$$
$$D^1 f(z) = Df(z) = zf'(z), \quad (1.3)$$

and

$$D^nf(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}). \quad (1.4)$$

With the above operator $D^n$, we say that a function $f(z)$ belonging to $A_j$ is in the class $A_j(n,m,a)$ if and only if

$$\text{Re} \left[ \frac{D^{n+m} f(z)}{D^n f(z)} \right] > a \quad (n,m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \quad (1.5)$$

for some $a (0 \leq a < 1)$, and for all $z \in U$. 
We note that $A_1(0,1,a) = S^*(a)$ is the class of starlike functions of order $a$, $A_1(1,1,a) = K(a)$ is the class of convex functions of order $a$, and that $A_1(n,1,a) = S_n(a)$ is the class of functions defined by Salagean [1].

Let $T_j$ denote the subclass of $A_j$ consisting of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0; j \in \mathbb{N}).$$

Further, we define the class $T_j(n,m,a)$ by

$$T_j(n,m,a) = A_j(n,m,a) \cap T_j.$$  

Then we observe that $T_1(0,1,a) = T^*(a)$ is the subclass of starlike functions of order $a$ (Silverman [2]), $T_1(1,1,a) = C(a)$ is the subclass of convex functions of order $a$ (Silverman [2]), and that $T_j(0,1,a)$ and $T_j(1,1,a)$ are the classes defined by Chatterjea [3].

2. DISTORTION THEOREMS.

We begin with the statement and the proof of the following result.

**Lemma 1.** Let the function $f(z)$ be defined by (1.6) with $j = 1$. Then $f(z) \in T_1(n,m,a)$ if and only if

$$\sum_{k=2}^{\infty} \frac{k^n (k^m - a) a_k}{k^m} \leq 1 - a$$

for $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, and $0 \leq a < 1$. The result is sharp.

**Proof.** Assume that the inequality (2.1) holds and let $|z| = 1$. Then we have

$$\left| \frac{D^{n+m} f(z)}{D^n f(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} k^n (k^m - 1) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k |z|^{k-1}}$$

$$= \frac{\sum_{k=2}^{\infty} k^n (k^m - 1) a_k}{1 - \sum_{k=2}^{\infty} k^n a_k}$$

$$\leq 1 - a$$

which implies (1.5). Thus it follows from this fact that $f(z) \in T_1(n,m,a)$.

Conversely, assume that the function $f(z)$ is in the class $T_1(n,m,a)$. Then

$$\text{Re} \left( \frac{D^{n+m} f(z)}{D^n f(z)} \right) = \text{Re} \left( \frac{1 - \sum_{k=2}^{\infty} k^n a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k z^{k-1}} \right)$$

$$> \alpha$$

(2.3)
for \( z \in \mathbb{U} \). Choose values of \( z \) on the real axis so that \( \frac{D^{n+m}f(z)}{D^n f(z)} \) is real. Upon clearing the denominator in (2.3) and letting \( z \) through real values, we obtain

\[
1 - \sum_{k=2}^{\infty} k^{n+m} a_k \geq \alpha (1 - \sum_{k=2}^{\infty} k^n a_k) \tag{2.4}
\]

which gives (2.1). The result is sharp with the extremal function \( f(z) \) defined by

\[
f(z) = z - \frac{1 - \alpha}{k^n(k^m - \alpha)} \quad (k \geq 2) \tag{2.5}
\]

REMARK I. In view of Lemma 1, \( T_1(n,m,a) \) when \( n \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \) is the subclass of \( T^*(a) \) introduced by Silverman [2], and \( T_1(n,m,a) \) when \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \) is the subclass of \( C(a) \) introduced by Silverman [2].

With the aid of Lemma 1, we prove

THEOREM 1. Let the function \( f(z) \) be defined by (1.6). Then \( f(z) \in T_j(n,m,a) \) if and only if

\[
\sum_{k=j+1}^{\infty} k^n(k^m - a)a_k \leq 1 - \alpha \tag{2.6}
\]

for \( n \in \mathbb{N}_0, m \in \mathbb{N}_0 \) and \( 0 \leq \alpha < 1 \). The result is sharp for the function

\[
f(z) = z - \frac{1 - \alpha}{k^n(k^m - \alpha)} \quad (k \geq j + 1). \tag{2.7}
\]

PROOF. Putting \( a_k = 0 \) \((k = 2, 3, 4, \ldots, j)\) in Lemma 1, we can prove the assertion of Theorem 1.

COROLLARY 1. Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then

\[
a_k \leq \frac{1 - \alpha}{k^n(k^m - \alpha)} \quad (k \geq j + 1). \tag{2.8}
\]

The equality in (2.8) is attained for the function \( f(z) \) given by (2.7).

COROLLARY 2. \( T_j(n+1,m,a) \subset T_j(n,m,a) \) and \( T_j(n,m+1,a) \subset T_j(n,m,a) \).

REMARK 2. Taking \((j,n,m) = (1,0,1)\) and \((j,n,m) = (1,1,1)\) in Theorem 1, we have the corresponding results by Silverman [2]. Taking \((j,n,m) = (j,0,1)\) and \((j,n,m) = (1,1,1)\) in Theorem 1, we have the corresponding results by Chatterjea [3].

THEOREM 2. Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then

\[
|D^i f(z)| \geq |z| - \frac{1 - \alpha}{(j + 1)^{-1}((j + 1)^m - \alpha)} |z|^{j+1} \tag{2.9}
\]

and

\[
|D^i f(z)| \leq |z| + \frac{1 - \alpha}{(j + 1)^{-1}((j + 1)^m - \alpha)} |z|^{j+1} \tag{2.10}
\]

for \( z \in \mathbb{U} \), where \( 0 \leq i \leq n \). The equalities in (2.9) and (2.10) are attained for the
function \( f(z) \) given by

\[
f(z) = z - \frac{1 - \alpha}{(j + 1)^n((j + 1)^m - \alpha)} z^{j+1}
\]  

(2.11)

**Proof.** Note that \( f(z) \in T_j(n,m,\alpha) \) if and only if \( D^i f(z) \in T_j(n-i,m,\alpha) \), and that

\[
D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k.
\]

(2.12)

Using Theorem 1, we know that

\[
(j + 1)^{n-i}((j + 1)^m - \alpha) \sum_{k=j+1}^{\infty} k^i a_k \leq 1 - \alpha,
\]

(2.13)

that is, that

\[
\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{1 - \alpha}{(j + 1)^{n-1}((j + 1)^m - \alpha)}.
\]

(2.14)

It follows from (2.12) and (2.14) that

\[
|D^i f(z)| \geq |z| - \frac{1 - \alpha}{(j + 1)^{n-1}((j + 1)^m - \alpha)} |z|^{j+1}
\]

(2.15)

and

\[
|D^i f(z)| \leq |z| + \frac{1 - \alpha}{(j + 1)^{n-1}((j + 1)^m - \alpha)} |z|^{j+1}.
\]

(2.16)

Finally, we note that the equalities in (2.9) and (2.10) are attained for the function \( f(z) \) defined by

\[
D^i f(z) = z - \frac{1 - \alpha}{(j + 1)^{n-i}((j + 1)^m - \alpha)} z^{j+1}.
\]

(2.17)

This completes the proof of Theorem 2.

**Corollary 3.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,\alpha) \). Then

\[
|f(z)| \geq |z| - \frac{1 - \alpha}{(j + 1)^n((j + 1)^m - \alpha)} |z|^{j+1}
\]

(2.18)

and

\[
|f(z)| \leq |z| + \frac{1 - \alpha}{(j + 1)^n((j + 1)^m - \alpha)} |z|^{j+1}
\]

(2.19)

for \( z \in \mathbb{U} \). The equalities in (2.18) and (2.19) are attained for the function \( f(z) \) given by (2.11).

**Proof.** Taking \( i = 0 \) in Theorem 2, we can easily show (2.18) and (2.19).
Corollary 4. Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then
\[
|f'(z)| \geq 1 - \frac{1 - \alpha}{(j + 1)^{n-1}(j + 1)^m - \alpha} |z|^j
\] (2.20)
and
\[
|f'(z)| \leq 1 + \frac{1 - \alpha}{(j + 1)^{n-1}(j + 1)^m - \alpha} |z|^j
\] (2.21)
for \( z \in U \). The equalities in (2.20) and (2.21) are attained for the function \( f(z) \) given by (2.11).

Proof. Note that \( Df(z) = zf'(z) \). Hence, making \( i = 1 \) in Theorem 2, we have the corollary.

Remark 3. Taking \( (j,n,m) = (1,0,1) \) and \( (j,n,m) = (1,1,1) \) in Corollary 3 and Corollary 4, we have distortion theorems due to Silverman [2].

3. Distortion Theorems for Fractional Calculus.

In this section, we use the following definitions of fractional calculus by Owa [4].

Definition 1. The fractional integral of order \( \lambda \) is defined by
\[
D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z - \xi)^{1-\lambda}} d\xi
\] (3.1)
where \( \lambda > 0 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin and the multiplicity of \( (z - \xi)^{\lambda-1} \) is removed by requiring \( \log(z - \xi) \) to be real when \( (z - \xi) > 0 \).

Definition 2. The fractional derivative of order \( \lambda \) is defined by
\[
D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z - \xi)^{1+\lambda}} d\xi,
\] (3.2)
where \( 0 < \lambda < 1 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin and the multiplicity of \( (z - \xi)^{-\lambda} \) is removed by requiring \( \log(z - \xi) \) to be real when \( (z - \xi) > 0 \).

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order \( (n + \lambda) \) is defined by
\[
D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z)
\] (3.3)
where \( 0 \leq \lambda < 1 \) and \( n \in \mathbb{N}_0 = \{0,1,2,3,\ldots\} \).

Theorem 3. Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then
\[
|D^{-\lambda}_z(D^4_1 f(z))| \geq \frac{|z|^{1+\lambda}}{\Gamma(2 + \lambda) \Gamma(1 + 2)\Gamma(2 + \lambda)(1 - \alpha)} \left( 1 - \frac{\Gamma(1 + 2)\Gamma(2 + \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 + \lambda)(j + 1)^{n-1}(j + 1)^m - \alpha} |z|^j \right)
\] (3.4)
and
\[ |D_z^{-\lambda}(D^i f(z))| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left( 1 + \frac{\Gamma(j+2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-1}((j+1)^m-\alpha)} |z|^j \right) \] (3.5)

for \( \lambda > 0, 0 < i \leq n, \) and \( z \in U. \) The equalities in (3.4) and (3.5) are attained for the function \( f(z) \) given by (2.11).

**PROOF.** It is easy to see that
\[ \Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z)) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} k^ia_kz^k. \] (3.6)

Since the function
\[ \phi(k) = \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} \quad (k \geq j+1) \] (3.7)
is decreasing in \( k, \) we have
\[ 0 < \phi(k) \leq \phi(j+1) = \frac{\Gamma(j+2)\Gamma(2+\lambda)}{\Gamma(j+2+\lambda)}. \] (3.8)

Therefore, by using (2.14) and (3.8), we can see that
\[ |\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z))| \geq |z| - \phi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^ia_k \]
\[ \geq |z| - \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-1}((j+1)^m-\alpha)} |z|^{j+1} \] (3.9)

which implies (3.4), and that
\[ |\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^i f(z))| \leq |z| + \phi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^ia_k \]
\[ \leq |z| + \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-1}((j+1)^m-\alpha)} |z|^{j+1} \] (3.10)

which shows (3.5). Furthermore, note that the equalities in (3.4) and (3.5) are attained for the function \( f(z) \) defined by
\[ D_z^{-\lambda}(D^i f(z)) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left( 1 - \frac{\Gamma(j+2)\Gamma(2+\lambda)\cdot(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-1}((j+1)^m-\alpha)} z^j \right) \] (3.11)
or (2.17). Thus we complete the assertion of Theorem 3.

Taking \( i = 0 \) in Theorem 3, we have

**COROLLARY 5.** Let the function \( f(z) \) by (1.6) be in the class \( T_j(n,m,\alpha). \) Then
\[ |D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2 + \lambda)} \left( 1 - \frac{\Gamma(1 + 2)\Gamma(2 + \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 + \lambda)(j + 1)^n((j + 1)^m - \alpha)}|z|^j \right) \]  \hspace{1cm} (3.12)

and

\[ |D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2 + \lambda)} \left( 1 + \frac{\Gamma(1 + 2)\Gamma(2 + \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 + \lambda)(j + 1)^n((j + 1)^m - \alpha)}|z|^j \right) \]  \hspace{1cm} (3.13)

for \( \lambda > 0 \) and \( z \in U \). The equalities in (3.12) and (3.13) are attained for the function \( f(z) \) given by (2.11).

Finally, we prove

**THEOREM 4.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then

\[ |D_z^{\lambda} (D^1 f(z))| \geq \frac{|z|^{1-\lambda}}{\Gamma(2 - \lambda)} \left( 1 - \frac{\Gamma(1 + 1)\Gamma(2 - \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 - \lambda)(j + 1)^{n-1}((j + 1)^m - \alpha)}|z|^j \right) \]  \hspace{1cm} (3.14)

and

\[ |D_z^{\lambda} (D^1 f(z))| \leq \frac{|z|^{1-\lambda}}{\Gamma(2 - \lambda)} \left( 1 + \frac{\Gamma(1 + 1)\Gamma(2 - \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 - \lambda)(j + 1)^{n-1}((j + 1)^m - \alpha)}|z|^j \right) \]  \hspace{1cm} (3.15)

for \( 0 \leq \lambda < 1, 0 \leq i \leq n - 1, \) and \( z \in U \).

The equalities in (3.14) and (3.15) are attained for the function \( f(z) \) given by (2.11).

**PROOF.** A simple computation gives that

\[ \Gamma(2 - \lambda)z^\lambda D_z^{\lambda} (D^1 f(z)) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k + 1)\Gamma(2 - \lambda)k!}{\Gamma(k + 1 - \lambda)} a_k z^k. \]  \hspace{1cm} (3.16)

Note that the function

\[ \psi(k) = \frac{\Gamma(k)\Gamma(2 - \lambda)}{\Gamma(k + 1 - \lambda)} \quad (k \geq j + 1) \]  \hspace{1cm} (3.17)

is decreasing in \( k \). It follows from this fact that

\[ 0 < \psi(k) \leq \psi(j + 1) = \frac{\Gamma(j + 1)\Gamma(2 - \lambda)}{\Gamma(j + 2 - \lambda)}. \]  \hspace{1cm} (3.18)

Consequently, with the aid of (2.14) and (3.18), we have

\[ |\Gamma(2 - \lambda)z^\lambda D_z^{\lambda} (D^1 f(z))| \geq |z| - \psi(j + 1)|z|^{j+1} \sum_{k=j+1}^{\infty} k! a_k \]  \hspace{1cm} (3.19)

and

\[ |\Gamma(2 - \lambda)z^\lambda D_z^{\lambda} (D^1 f(z))| \leq |z| + |\psi(j + 1)|z|^{j+1} \sum_{k=j+1}^{\infty} k! a_k \]
Thus (3.14) and (3.15) follow from (3.19) and (3.20), respectively. Further, since the equalities in (3.19) and (3.20) are attained for the function $f(z)$ defined by

$$z F(j + 1)F(2- A)(1D%) F(2 ) F(j + 2- %)(j + 1)n-l{(j + I)m- } \tag{3.21}$$

that is, by (2.17), this completes the proof of Theorem 4.

Making $i = 0$ in Theorem 4, we have

**Corollary 6.** Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,=)$. Then

$$|D^\lambda z f(z)| \geq \frac{|z|^{1-\lambda}}{\Gamma(2- \lambda)} \left| 1 - \frac{\Gamma(j + 1)\Gamma(2 - \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 - \lambda)(j + 1)\Gamma(j + n-1{(j + I)m- } \right| z^j} \tag{3.22}$$

and

$$|D^\lambda z f(z)| \leq \frac{|z|^{1-\lambda}}{\Gamma(2- \lambda)} \left| 1 + \frac{\Gamma(j + 1)\Gamma(2 - \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 - \lambda)(j + 1)\Gamma(j + n-1{(j + I)m- } \right| z^j} \tag{3.23}$$

for $0 \leq \lambda < 1$ and $z \subseteq U$. the equalities in (3.22) and (3.23) are attained for the function $f(z)$ given by (2.11).

**Acknowledgement.** The author wishes to thank the referee for his helpfull comments.

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