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TRANSFER SCATTERING MATRIX OF NON-UNIFORM SURFACE ACOUSTIC WAVE TRANSDUCERS

N.C. DEBNATH and T.ROY

Department of Physics
Jadavpur University
Calcutta - 700 032, INDIA

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ABSTRACT. This paper is concerned with a general mathematical theory for finding the admittance matrix of a three-port non-uniform surface acoustic wave (SAW) network characterized by n unequal hybrid sections. The SAW interdigital transducer and its various circuit model representations are presented in some detail. The Transfer scattering matrix of a transducer consisting of N non-uniform sections modeled through the hybrid equivalent circuit is discussed. General expression of the scattering matrix elements for a N-section SAW network is included. Based upon hybrid equivalent circuit model of one electrode section, explicit formulas for the scattering and transfer scattering matrices of a SAW transducer are obtained. Expressions of the transfer scattering matrix elements for the N-section crossed-field and in-line model of SAW transducers are also derived as special cases. The matrix elements are computed in terms of complex frequency and thus allow for transient response determinations. It is shown that the general forms presented here for the matrix elements are suitable for the computer aided design of SAW transducers.

KEYWORDS AND PHRASES. Surface Acoustic waves, Interdigital transducers, scattering and transfer scattering matrices, Hybrid in-line and crossed-field models.
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1. INTRODUCTION

Surface Acoustic Wave (SAW) devices are essentially concerned with the use of the features of elastic waves of high frequency. These elastic waves are generally guided along the interface between two media, one of which is solid while the other one can be air or vacuum. The solid medium is generally taken as piezo-electric. These waves are non-dispersive in nature and have displacements which rapidly decay in exponential form away from the free surface and into the solid. Thus, almost all the wave energy is confined within a depth equal to one wave length. The most important feature of a surface elastic wave is its extremely small velocity, approximately 10^{-5} times the velocity of electromagnetic waves; hence the name "acoustic". In view of several important features of surface acoustic waves, there is a tremendous impact of these waves on the design of different kinds of delay lines and filters.

The principle factor in the emergence of the surface acoustic wave is due to the development of the interdigital transducer (IDT), first introduced in the literature by White and Voltmer [1]. The transducer is regarded as a fundamental device for efficient generation and reception of the acoustic wave, and forms the basis for most of the wide variety of surface acoustic wave devices.

It has been shown [2] that circuit model representations can be the key for the development and design of SAW devices. Especially, starting from Mason's characterizations [3], an equivalent circuit representation [2,4,5] of a transducer has been found to be very successful in characterizing the processing of surface waves. In these references the transfer characteristics and the input admittance at the electrical port of the interdigital transducers, have been determined [5,6]. Smith et al [5], and Gerard [6] have calculated reflection coefficients, but only at the synchronous frequency. Unfortunately, though, there is still not available a good circuit theory of SAW devices. Recently [7,8,9] a formation to obtain the scattering matrices of SAW devices has been outlined. This paper applies these techniques to derive the transfer scattering matrix of N-section SAW tansducers which can have equal or unequal sections characterized by the hybrid equivalent circuit. This allows for representation through crossed-field or in-line model or anything in between, as well as extensions to more elaborate models.

Section 2 is devoted to a brief introduction to the SAW interdigital transducer, and its various circuit model representations. In section 3, a general mathematical theory is developed for finding the admittance matrix of a three-port non-uniform SAW network characterized by N unequal hybrid sections. General expressions of the scattering matrix elements for an N-section SAW network is included in section 4. Finally, sections 5 and 6 deal with the development of explicit formulas for the scattering and transfer scattering matrices of a SAW transducer based upon the hybrid equivalent circuit model of one electrode section. Expressions of the transfer scattering matrix elements for the N-section crossed-field and in-line model of SAW transducers are also derived as special cases. The matrix elements are computed in terms of complex frequency and thus allow for transient response determinations. The general forms presented for the matrix elements are suitable for the computer-aided design of SAW transducers.

2. SAW INTERDIGITAL TRANSDUCER

The most fundamental componenet in the surface acoustic wave field is the interdigital transducer. Its great importance lies in its dual role as a common converter between electrical signals and surface acoustic waves and as a building block for wave shaping in various surface wave devices. In view of its inherent versatility, the interdigital transducer is employed by all surface wave components.

An interdigital transducer consists of a series of interleaved metal electrodes deposited on a piezoelectric substrate. The width of the electrodes is equal to the width of the interelectrode gaps. So the transducer is a two dimensional device with alternate electrodes interconnected. Upon application of a voltage, an electric field distribution with the spatial period of the electrodes is produced between the electrodes.

Figure 1 shows a top view of a typical N section SAW transducer. Used as a transmitter, a voltage source is applied between the upper and lower fingers. These fingers generate mechanical motion, an acoustic wave, in the direction shown by virtue of being placed on piezoelectric material. The transducer can also be used as a receiver since the acoustic wave incident on the piezoelectric material generates an electric field which can be detected as a voltage between the upper and lower fingers.

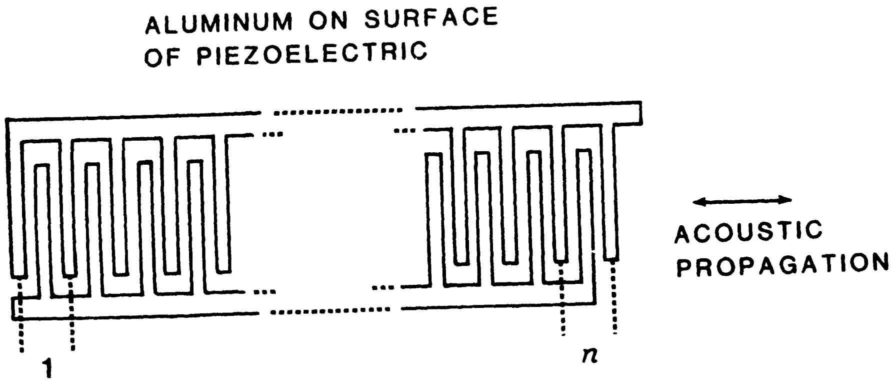

Figure 1. Schematic diagram of a SAW Transducer

Since a single pair of electrodes is not very efficient in generating acoustic waves, several pairs of electrodes are generally employed in the form of an interdigitating pattern. Each pair of the electrodes generates the so called Rayleigh waves. The interdigital transducer is designed in a special manner so that the separately excited Rayleigh waves reinforce one another and produce a large acoustic signal of practical use. This is successfully achieved by selecting the spacing between each pair of fingers such that the Rayleigh waves can travel that distance in exactly the time required for the generating signal to repeat itself. If the frequency of the waves is changed from the ideal value, the individual excitations due to every pair of fingers have a tendency to cancel one another. Therefore a long transducer is used to avoid such cancellation. In other words, a transducer containing a large number of electrode pairs would be efficient for generating and receiving electrical signals over only a narrow frequency range, and can be utilized as a filter to sort out signals of one frequency from signals of another frequency. Conversely, a short transducer can be employed to generate signals over a wider frequency range. Such filters are found to have useful applications in communication systems.

Figure 2(a) shows a cross section of the transducer, through the piezoelectric material by a plane in the direction of acoustic propagation, to illustrate the electric field pattern. The field is seen to have two components, one "crossed-field in which the applied electric field is perpendicular to the acoustic wave propagation direction and one "in-line" which is characterized by parallel electric field and wave propagation vector. These representations are illustrated in Figures 2(b) and 2(c), and commonly referred to as the crossed-field model and in-line field model of a transducer section, respectively. The transducer or any section of it is therefore seen to be representable by a three-port where one port is electrical (for the

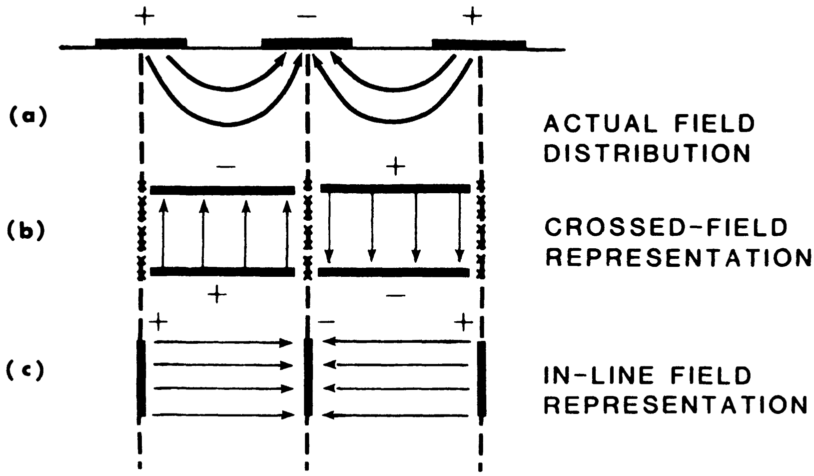

Figure 2. Side view of the Transducer showing electric field patterns

transducer fingers) and two ports are mechanical (for the two acoustical ends). Further, it has been found that such a three-port SAW transducer section can be represented by the Mason equivalent circuit [3]. The equivalent circuit of the entire transducer is then found by cascading the mechanical ports of the circuits of individual sections, and gives the relationship between input signals at the three-ports (one electric and two acoustic).

It has been shown [5] that both the crossed-field and in-line models can predict transducer performance relatively accurately, and they can be used to calculate three-port admittance matrix of a complete transducer in a very simple and straightforward manner. However, for our purpose we shall consider the more general model of a SAW transducer section, commonly known as "hybrid" model of a three-port SAW section. The hybrid model can also be represented by a three-port equivalent circuit in which an additional parameter α, denoting the crossed-field to in-line field coupling coefficient and $0 \leq \alpha \leq 1$, has been introduced. The hybrid model reduces to the crossed-field model when $\alpha=0$, and to the in-line model when $\alpha=1$, with the intermediate values of α giving different weights to the crossed and in-line fields present in the actual field distribution. Therefore, the hybrid model allows for representations of fields which are combinations of those of crossed-field and in-line models. This hybrid model has found some success in the past [10] and we believe values of $\alpha \simeq 0.5$ most reasonable.

3. ADMITTANCE MATRIX FOR A THREE-PORT NON-UNIFORM SAW TRANSDUCER

We consider a three-port SAW transducer section, as shown in Figure 3, with two identical mechanical ports of variables

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \text{force applied},$$

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \text{velocity of material},$$

where F_1 and F_2 denote the input and the output forces respectively, and one electric port of variables V_3= voltage and I_3= current applied to the interdigital electrodes.

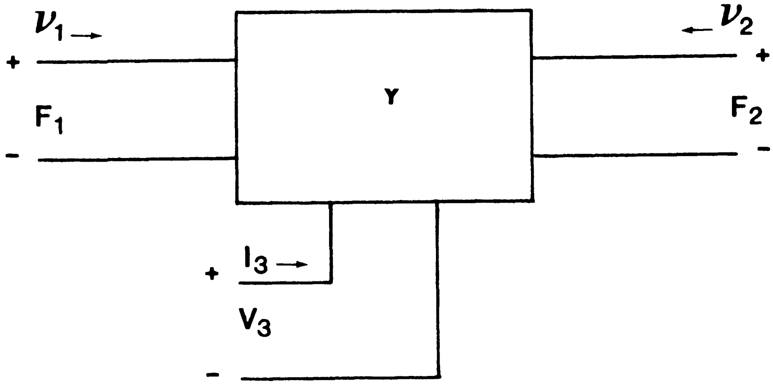

Figure 3. Block-diagram of a three-port Model

Then, for the hybrid model, we can write the admittance matrix description as

$$\begin{bmatrix} V_1 \\ V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{12} & y_{11} & y_{13} \\ y_{13} & y_{13} & y_{33} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ V_3 \end{bmatrix} \tag{3.1}$$

where $y_{22} = y_{11}$ and $y_{23} = y_{13}$ by symmetry. y is symmetric by the reciprocity of the equivalent circuits.

We now connect N of these three-port sections in cascade at the mechanical ports and in parallel, with every other one reversed, at the electrical ports as shown in Figure 4.

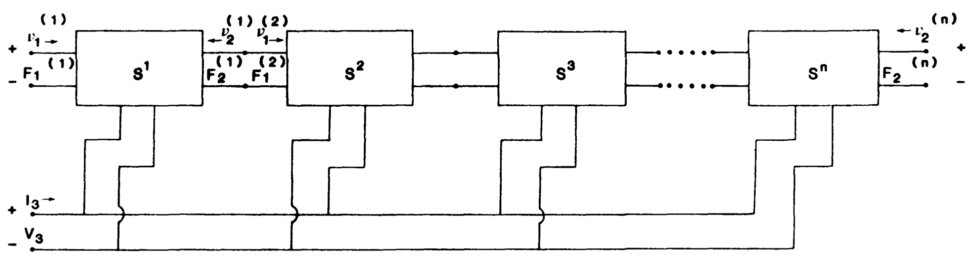

Figure 4. Cascaded one-electrode sections for a SAW transducer

We consider all basic transducer sections to be described by different parameters. Thus for the ith section we have

$$\begin{bmatrix} V_1^{(i)} \\ V_2^{(i)} \\ I_3^{(i)} \end{bmatrix} = \begin{bmatrix} y_{11}^{(i)} & y_{12}^{(i)} & y_{13}^{(i)} \\ y_{12}^{(i)} & y_{11}^{(i)} & y_{13}^{(i)} \\ y_{13}^{(i)} & y_{13}^{(i)} & y_{33}^{(i)} \end{bmatrix} \begin{bmatrix} F_1^{(i)} \\ F_2^{(i)} \\ V_3^{(i)} \end{bmatrix} \tag{3.2}$$

For mechanical ports the interconnection laws, with $i > 1$, can be expressed as

$$\begin{aligned} F_1^{(i)} &= F_2^{(i-1)} \\ V_1^{(i)} &= V_2^{(i-1)} \end{aligned} \tag{3.3}$$

where

$$\left.\begin{aligned} F_1^{(1)} &= F_{in} = F_{\text{left boundary}}, & V_1^{(1)} &= V_{in} \\ F_2^{(N)} &= F_{out} = F_{\text{right boundary}}, & V_2^{(N)} &= V_{out} \end{aligned}\right\} \tag{3.4}$$

Also for electrical ports with $i > 1$, the interconnection laws are

$$\left.\begin{aligned} V_3^{(i)} &= (-1)^{i-1}\, V_3 \\ I_3 &= \sum_{i=1}^{N} (-1)^{i-1}\, I_3^{(i)} \end{aligned}\right\} \tag{3.5}$$

The resulting network is a three-port, and can conveniently be described by a single section as shown in Figure 5. In order to find the interconnected network three-port descriptions, we convert to a hybrid reverse chain matrix, Φ_R, type of description.

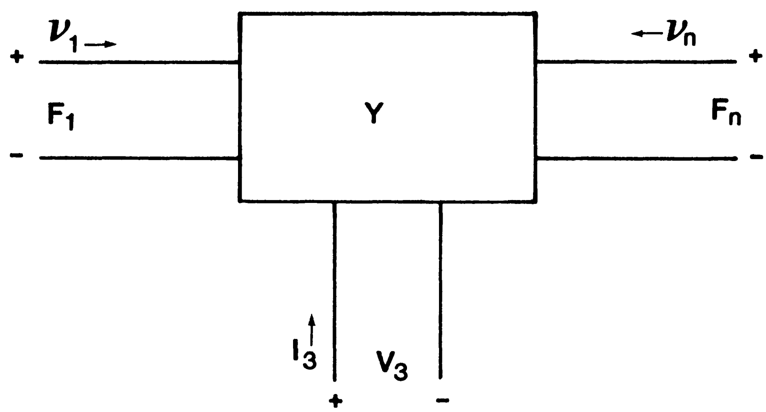

Figure 5. Combined Network Structure

From Equation (3.2), it follows that

$$\begin{bmatrix} F_2^{(i)} \\ -V_2^{(i)} \end{bmatrix} = \Psi_R^{(i)} \begin{bmatrix} F_1^{(i)} \\ V_1^{(i)} \end{bmatrix} + A^{(i)} V_3 \tag{3.6}$$

where

$$\Psi_R^{(i)} = -\frac{1}{y_{12}^{(i)}} \begin{bmatrix} y_{11}^{(i)} & -1 \\ [(y_{12}^{(i)})^2 - (y_{11}^{(i)})^2] & y_{11}^{(i)} \end{bmatrix} \tag{3.7}$$

$$A^{(i)} = \frac{-(-1)^{i-1} y_{13}^{(i)}}{y_{12}^{(i)}} \begin{bmatrix} 1 \\ (y_{12}^{(i)} - y_{11}^{(i)}) \end{bmatrix} \tag{3.8}$$

and

$$I_3^{(i)} = -\frac{y_{13}^{(i)}}{y_{12}^{(i)}} [y_{11}^{(i)} - y_{12}^{(i)},\ 1] \begin{bmatrix} F_1^{(i)} \\ V_1^{(i)} \end{bmatrix}$$

$$+ \frac{(-1)^{i-1}}{y_{12}^{(i)}} [y_{33}^{(i)} \quad y_{12}^{(i)} - (y_{13}^{(i)})^2]\ V_3 \tag{3.9}$$

Equation (3.6) is the key for this analysis. Using the mechanical interconnections, we obtain

$$\begin{bmatrix} F_2^{(1)} \\ -V_2^{(1)} \end{bmatrix} = \psi_R^{(1)} \begin{bmatrix} F_{in} \\ V_{in} \end{bmatrix} + A^{(1)} V_3 , \tag{3.10}$$

$$\begin{bmatrix} F_2^{(2)} \\ -V_2^{(2)} \end{bmatrix} = \psi_R^{(2)} \begin{bmatrix} F_1^{(2)} \\ V_1^{(2)} \end{bmatrix} + A^{(2)} V_3 \tag{3.11}$$

or

$$\begin{bmatrix} F_2^{(2)} \\ -V_2^{(2)} \end{bmatrix} = \psi_R^{(2)} \quad \psi_R^{(1)} \begin{bmatrix} F_{in} \\ V_{in} \end{bmatrix} + \{\psi_R^{(2)} A^{(1)} + A^{(2)}\}\ V_3 \tag{3.12}$$

In general for N sections

$$\begin{bmatrix} F_{out} \\ -V_{out} \end{bmatrix} = \begin{bmatrix} F_2^{(N)} \\ -V_2^{(N)} \end{bmatrix} = \psi_R^{(N)} \begin{bmatrix} F_1^{(N)} \\ -V_1^{(N)} \end{bmatrix} + A^{(N)} V_3$$

$$= \psi_R^{(N)} \psi_R^{(N-1)} \ldots \psi_R^{(2)} \psi_R^{(1)} \begin{bmatrix} F_{in} \\ V_{in} \end{bmatrix}$$

$$+ \{A^{(N)} + \psi_R^{(N)} A^{(N-1)} + \ldots\ldots + \psi_R^{(N)} \psi_R^{(N-1)} \ldots \quad (3.13)$$

$$\ldots\ldots \psi_R^{(3)} A^{(2)} + \psi_R^{(N)} \psi_R^{(N-1)} \ldots\ldots \psi_R^{(3)} \psi_R^{(2)} A^{(1)}\}$$

Thus for the entire cascaded system

$$\begin{bmatrix} F_{out} \\ -V_{out} \end{bmatrix} = \Phi_R \begin{bmatrix} F_{in} \\ V_{in} \end{bmatrix} + A\, V_3 \quad (3.14)$$

where

$$\Phi_R = \prod_{i=1}^{N}{}_R \; \psi_R^{(i)}, \quad (3.15)$$

$$A = \sum_{i=1}^{N} \left(\prod_{k=i}^{N-1}{}_R \; \psi_R^{(k+1)}\right) A^{(i)}, \quad (3.16)$$

and

$$\prod_{k=N}^{N-1} = \text{identity.}$$

These general Equations (3.14) to (3.16) will be used to derive the admittance matrix for the transducer consisting of N non-uniform basic sections.

Explicit expansion of the matrix Equation (3.14) yields the upper two rows of the admittance matrix as per (3.1).

$$\begin{bmatrix} V_{in} \\ V_{out} \end{bmatrix} = \begin{bmatrix} -\Phi_{R_{12}}^{-1} \Phi_{R_{11}} & \Phi_{R_{12}}^{-1} & -\Phi_{R_{12}}^{-1} A_{11} \\ (-\Phi_{R_{21}} + \Phi_{R_{22}} \Phi_{R_{12}}^{-1} \Phi_{R_{11}}) & -\Phi_{R_{22}} \Phi_{R_{12}}^{-1} & (\Phi_{R_{22}} \Phi_{R_{12}}^{-1} A_{11} - A_{21}) \end{bmatrix} \begin{bmatrix} F_{in} \\ F_{out} \\ V_3 \end{bmatrix} \quad (3.17)$$

where $\Phi_{R_{11}}, \Phi_{R_{12}}, \Phi_{R_{22}}$ etc. represent the corresponding elements of the final matrix Φ_R for the entire cascaded network.

In order to obtain the third row of the admittance matrix Y, we begin with Equation (3.16) which can be rewritten in the form

$$A = \sum_{i=1}^{N-1} \left(\prod_{j=i+1}^{N} \psi_R^{(j)} \right) A^{(i)} + A^{(N)} \tag{3.18}$$

and

Using the explicit value of $A^{(i)}$ given by Equation (3.8), Equation (3.18) becomes

$$A = \sum_{i=1}^{N-1} \left\{ \prod_{j=i+1}^{N} \psi_R^{(j)} \left[\frac{(-1)^i \, y_{13}^{(i)}}{y_{12}^{(i)}} \right] \begin{bmatrix} 1 \\ y_{12}^{(i)} - y_{11}^{(i)} \end{bmatrix} \right\} + \frac{(-1)^N \, y_{13}^{(N)}}{y_{12}^{(N)}} \begin{bmatrix} 1 \\ y_{12}^{(N)} - y_{11}^{(N)} \end{bmatrix} \tag{3.19}$$

We have from Equation (3.5)

$$I_3 = \sum_{i=1}^{N} (-1)^{i-1} I_3^{(i)}$$

$$= \sum_{i=1}^{N} (-1)^{i-1} y_{13}^{(i)} [F_1^{(i)} + F_2^{(i)}] + \sum_{i=1}^{N} y_{33}^{(i)} V_3 \tag{3.20}$$

where

$$F_1^{(i)} = F_2^{(i-1)} \tag{3.21}$$

$$\begin{bmatrix} F_2^{(i)} \\ -V_2^{(i)} \end{bmatrix} = \prod_{j=1}^{i} \psi_R^{(j)} \begin{bmatrix} F_{in} \\ V_{in} \end{bmatrix} + \sum_{j=1}^{i} \prod_{k=j}^{i-1} \psi_R^{(k+1)} A^{(j)} V_3 , \tag{3.22}$$

and

$$\begin{bmatrix} F_1^{(i)} \\ -V_2^{(i-1)} \end{bmatrix} = \prod_{j=1}^{i-1} \psi_R^{(j)} \begin{bmatrix} F_{in} \\ V_{in} \end{bmatrix} + \sum_{j=1}^{i-1} \prod_{k=j}^{i-2} \psi_R^{(k+1)} A^{(j)} V_3 \tag{3.23}$$

for which the (1.1) entries are to be summed and the expression for V_{in} substituted. The general Equations (3.22) and (3.23) are also rewritten as

$$\begin{bmatrix} F_2^{(i)} \\ -V_2^{(i)} \end{bmatrix} = C^{(i)} \begin{bmatrix} F_{in} \\ V_{in} \end{bmatrix} + D^{(i)} V_3 \tag{3.24}$$

and

$$\begin{bmatrix} F_1^{(i)} \\ \\ -V_2^{(i-1)} \end{bmatrix} = C^{(i-1)} \begin{bmatrix} F_{in} \\ \\ V_{in} \end{bmatrix} + D^{(i-1)} V_3 \tag{3.25}$$

where the matrices $C^{(\ell)}$ and $D^{(\ell)}$ are defined as

$$C^{(\ell)} = \prod_{j=1}^{\ell} \psi_R^{(j)}; \quad D^{(\ell)} = \sum_{j=1}^{\ell} \prod_{k=j}^{\ell-1} \psi_R^{(k+1)} \; A^{(j)} \tag{3.26}$$

Expansion of the matrix Equations (3.24) and (3.25) gives

$$F_2^{(i)} = [C_{11}^{(i)} F_{in} + C_{12}^{(i)} \; V_{in}] + D_{11}^{(i)} \; V_3 \tag{3.27}$$

$$F_1^{(i)} = [C_{11}^{(i-1)} \; F_{in} + C_{12}^{(i-1)} \; V_{in}] + D_{11}^{(i-1)} \; V_3 \tag{3.28}$$

The use of the Equations (3.14), (3.27) and (3.28) into (3.20) yields the value of I_3 in its final form, and is given by

$$\begin{aligned} I_3 = \sum_{i=1}^{N} (-1)^{i-1} \; y_{13}(i) \; [\{ [C_{11}^{(i)} + C_{11}^{(i-1)}] - [C_{12}^{(i)} + C_{12}^{(i-1)}] \; \Phi_{R_{11}} \Phi_{R_{12}}^{-1} \} \; F_{in} \\ + [C_{12}^{(i)} + C_{12}^{(i-1)}] \; \Phi_{R_{12}}^{-1} \; F_{out} + \{[D_{11}^{(i)} + D_{11}^{(i-1)}] - [C_{12}^{(i)} + C_{12}^{(i-1)}] \; A_{11} \Phi_{R_{12}}^{-1}\} \; V_3] \\ + \sum_{i=1}^{N} y_{33}^{(i)} \; V_3 \end{aligned} \tag{3.29}$$

Therefore, the Equation (3.17) along with the Equation (3.29) gives the admittance matrix for a non-uniform SAW transducer in a very general form; they are written in component form as follows.

$$Y_{11} - \Phi_{R_{11}} \Phi_{R_{12}}^{-1} \tag{3.30}$$

$$Y_{12} = \Phi_{R_{12}}^{-1} \tag{3.31}$$

$$Y_{13} = -\Phi_{R_{12}}^{-1} A_{11} \tag{3.32}$$

$$Y_{21} = -\Phi_{R_{21}}^{-1} + \Phi_{R_{22}} \Phi_{R_{12}}^{-1} \Phi_{R_{11}} \tag{3.33}$$

$$Y_{22} = -\Phi_{R_{22}} \Phi_{R_{12}}^{-1} \tag{3.34}$$

$$Y_{23} = \Phi_{R_{22}} \Phi_{R_{12}}^{-1} \; A_{11} - A_{21} \tag{3.35}$$

$$Y_{31}=\sum_{i=1}^{N}(-1)^{i-1}y_{13}^{(i)}\{[C_{11}^{(i)}+C_{11}^{(i-1)}]-[C_{12}^{(i)}+C_{12}^{(i-1)}]\,\Phi_{R_{11}}\Phi_{R_{12}}^{-1}\} \tag{3.36}$$

$$Y_{32}=\sum_{i=1}^{N}(-1)^{i-1}\,y_{13}^{(i)}\;[C_{12}^{(i)}+C_{12}^{(i-1)}]\;\Phi_{R_{12}}^{-1} \tag{3.37}$$

$$Y_{33}=\sum_{i=1}^{N}[(-1)^{i-1}y_{13}^{(i)}\;\{[D_{11}^{(i)}+D_{11}^{(i-1)}]-[C_{12}^{(i)}+C_{12}^{(i-1)}]A_{11}\Phi_{R_{12}}^{-1}\}+y_{33}^{(i)}] \tag{3.38}$$

Since the resulting three-port is reciprocal, as an interconnection of reciprocal subnetworks, we know that $Y_{12}=Y_{21}$, $Y_{13}=Y_{31}$, and $Y_{23}=Y_{32}$.

The situation where all sections are equal is customary and mathematically much more tractable. For identical sections, it turns out [9] that

$$\Phi_R=(\psi_R^{(1)})^N=\frac{(-1)^N}{y_{12}^N}\begin{bmatrix} y_{11} & -1 \\ y_{12}^2-y_{11}^2 & y_{11}\end{bmatrix}^N \tag{3.39}$$

and

$$A=\sum_{k=0}^{N-1}(\psi_R^{(1)})^k\;(-1)^{N-k}\frac{y_{13}}{y_{12}}\begin{bmatrix}1\\ y_{12}-y_{11}\end{bmatrix} \tag{3.40}$$

where $N-i=k$.

By the use of induction, an explicit evaluation of matrix elements in these expressions gives

$$Y_{13}=y_{13},\; Y_{23}=(-1)^{N-1}y_{13}$$

where Y_{ij} are the matrix elements of the admittance matrix Y for the N cascaded sections, and y_{ij} are the matrix elements for the single electrode section. Also by reciprocity $Y_{31}=y_{13}$ and $Y_{32}=Y_{23}$ which can be verified by induction.

Therefore, the simplified form of the admittance matrix Y for a transducer consisting of N equal basic sections is given by

$$Y=\begin{bmatrix} -\Phi_{R_{12}}^{-1}\Phi_{R_{11}} & \Phi_{R_{12}}^{-1} & y_{13} \\ \Phi_{R_{12}}^{-1} & -\Phi_{R_{12}}^{-1}\Phi_{R_{11}} & (-1)^{N-1}y_{13} \\ y_{13} & (-1)^{N-1}y_{13} & Ny_{33}\end{bmatrix} \tag{3.41}$$

4. SCATTERING MATRIX FOR N SECTIONS

The scattering matrix of a three-port network characterized by its admittance matrix Y is given by [11]

$$S = \Pi_3 - 2Y\,(\Pi_3 + Y)^{-1} \tag{4.1}$$

where Π_3 is the 3×3 identity matrix.

Considering all the distinct elements of Y, the scattering matrix elements for a general three-port network consisting of N unequal sections are given by the following expressions.

$$\begin{aligned} S_{11} = \frac{1}{M} [& (1 + Y_{33})(1 - Y_{11} + Y_{22} - Y_{11}Y_{22} + Y_{12}Y_{21}) \\ & + Y_{13}\{Y_{31}(1 + Y_{22}) - Y_{21}Y_{32}\} \\ & + Y_{23}\{Y_{32}(Y_{11} - 1) - Y_{12}Y_{31}\}] \end{aligned} \tag{4.2}$$

$$S_{12} = -\frac{2}{M} [Y_{12}(1 + Y_{33}) - Y_{13}Y_{32})] \tag{4.3}$$

$$S_{13} = -\frac{2}{M} [Y_{13}(1 + Y_{22}) - Y_{12}Y_{23}] \tag{4.4}$$

$$S_{21} = -\frac{2}{M} [Y_{21}(1 + Y_{33}) - Y_{23}Y_{31}] \tag{4.5}$$

$$\begin{aligned} S_{22} = \frac{1}{M} [& (1 + Y_{33})(1 + Y_{11} - Y_{22} - Y_{11}Y_{22} + Y_{12}Y_{21}) \\ & + Y_{13}\{ Y_{31}(Y_{22} - 1) - Y_{21}Y_{32}\} \\ & + Y_{23}\{ Y_{32}(1 + Y_{11}) - Y_{12}Y_{31}\}] \end{aligned} \tag{4.6}$$

$$S_{23} = -\frac{2}{M} [Y_{23}(1 + Y_{11}) - Y_{13}Y_{21}] \tag{4.7}$$

$$S_{31} = -\frac{2}{M} [Y_{31}(1 + Y_{22}) - Y_{21}Y_{32}] \tag{4.8}$$

$$S_{32} = -\frac{2}{M} [Y_{32}(1 + Y_{11}) - Y_{12}Y_{31}] \tag{4.9}$$

$$\begin{aligned} S_{33} = \frac{1}{M} [& (1 - Y_{33})(1 + Y_{11} + Y_{22} + Y_{11}Y_{22} - Y_{12}Y_{21} \\ & + Y_{13}\{ Y_{31}(1 + Y_{22}) - Y_{21}Y_{32} \} \\ & + Y_{23}\{ Y_{32}(1 + Y_{11}) - Y_{12}Y_{31}\}] \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} M = \det(\Pi_3 + Y) = & (1 + Y_{33})[(1 + Y_{11})(1 + Y_{22}) - Y_{12}Y_{21}] \\ & - Y_{23}[Y_{32}(1 + Y_{11}) - Y_{12}Y_{31}] \\ & - Y_{13}[Y_{31}(1 - Y_{22}) - Y_{21}Y_{32}] \end{aligned} \tag{4.11}$$

For N identical sections, the Equations (4.2) to (4.11) can be further simplified. In particular, for the equal section case,

$Y_{11}= Y_{22}$, $Y_{12}= Y_{21}$, $Y_{13}= Y_{31}$, and $Y_{23}= Y_{32}= (-1)^{N-1}\ Y_{13}$.

Therefore S_{ij}'s take the following form

$$S_{11}= S_{22}= \frac{1}{M} [(1 + Y_{33})\ (1 - Y_{11}^2+ Y_{12}^2) + 2\ Y_{13}^2\ \{ Y_{11}+ (-1)^N\ Y_{12}\}] \tag{4.12}$$

$$S_{12}= S_{21}= - \frac{2}{M} [Y_{12}(1 + Y_{33}) + (-1)^N\ Y_{13}^2] \tag{4.13}$$

$$S_{13}= S_{31}= - \frac{2}{M} [Y_{13}\{1 + Y_{11}+ (-1)^N\ Y_{12}\}] \tag{4.14}$$

$$S_{23}= S_{32}= \frac{2}{M} [Y_{13}\{(-1)^N\ (1 + Y_{11}) + Y_{12}\}] = (-1)^{N-1} S_{13} \tag{4.15}$$

$$S_{33}= \frac{1}{M} [(1 - Y_{33})\ \{ (1 + Y_{11})^2- Y_{12}^2\} + 2\ Y_{13}^2\ \{ 1 + Y_{11}+ (-1)^N\ Y_{12}\}] \tag{4.16}$$

with

$$M = (1 + Y_{33})\ [(1 + Y_{11})^2 - Y_{12}^2] - 2\ Y_{13}^2\ [1 + Y_{11}+ (-1)^N\ Y_{12}] \tag{4.17}$$

5. HYBRID EQUIVALENT CIRCUIT AND ITS ADMITTANCE MATRIX

The three-port equivalent circuit of the hybrid model for a single electrode section of the transducer is shown in Figure 6. The stresses and particle velocities at the acoustic ports are represented by equivalent voltages and currents, respectively, to which they are numerically equal. A convenient method for describing the relationship between these currents and voltages is in terms of the admittance matrix, y_{ij}.

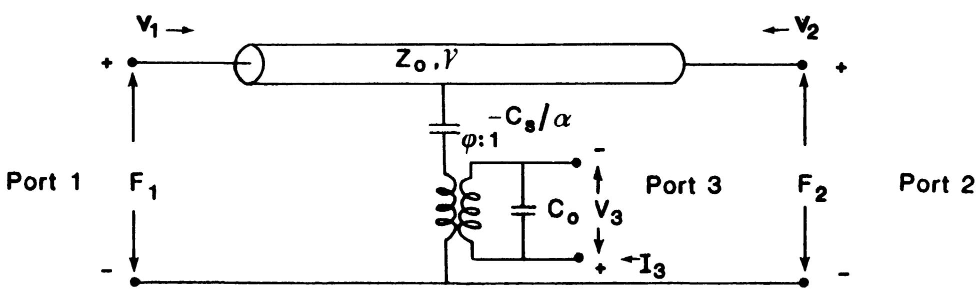

Figure 6. Hybrid Equivalent Circuit for one-electrode section

The standard network analysis [9] leads to the admittance matrix description of the hybrid model. The elements of the admittance matrix for the single electrode section are written as

$$y_{11} = Y_o \frac{\left(\operatorname{ctnh} \gamma p - \frac{\alpha}{C_s Z_o p}\right)}{\left(1 - \frac{2\alpha}{C_s Z_o p} \tanh \frac{\gamma p}{2}\right)} = y_{22} \tag{5.1}$$

$$y_{12} = - Y_o \frac{\left(\operatorname{csch} \gamma p - \frac{\alpha}{C_s Z_o p}\right)}{\left(1 - \frac{2\alpha}{C_s Z_o p} \tanh \frac{\gamma p}{2}\right)} = y_{21} \tag{5.2}$$

$$y_{13} = \phi Y_o \frac{\tanh \frac{\gamma p}{2}}{\left(1 - \frac{2\alpha}{C_s Z_o p} \tanh \frac{\gamma p}{2}\right)} = y_{23} = y_{31} = y_{32} \tag{5.3}$$

and

$$y_{33} = C_o p \left[1 + \frac{2\phi^2 Y_o}{C_o p} \frac{\tanh \frac{\gamma p}{2}}{\left(1 - \frac{2\alpha}{C_s Z_o p} \tanh \frac{\gamma p}{2}\right)}\right] \tag{5.4}$$

where

$-C_s = -\frac{Co}{\phi^2}$, negative series capacitance

$\phi = \left(\frac{k^2 C_o Z_o V_a}{\ell}\right)^{1/2}$, transformer turns ratio

$\gamma = \ell / V_a$

$p = \sigma + j\omega$

$\omega = 2\pi f$, f = frequency, $j = \sqrt{-1}$

k^2 = piezoelectric coupling coefficient

C_o = capacitance for a single section

Z_o = acoustic charactersitic impedance

V_a = velocity of acoustic propagation

ℓ = length of the single section, and

α = cross-field to in-line field coupling coefficient, $0 < \alpha < 1$.

Following the Equation (3.39) and after considerable manipulations, we find the admittance matrix for N equal sections of the hybrid model in the form below.

$$Y = \frac{Y_o}{D(p)} \begin{bmatrix} D^{\frac{1}{2}}(p)\ \text{ctnh}\ (N\delta(p)) & -D^{\frac{1}{2}}(p)\ \text{csch}(n\delta(p)) & \phi\ \tanh\ (\frac{\gamma p}{2}) \\ -D^{\frac{1}{2}}(p)\ \text{csch}\ (n\delta(p)) & D^{\frac{1}{2}}(p)\ \text{ctnh}(N\delta(p)) & (-1)^{N-1}\phi\ \tanh\ (\frac{\gamma p}{2}) \\ \phi\ \tanh\ (\frac{\gamma p}{2}) & (-1)^{N-1}\phi\ \tanh\ (\frac{\gamma p}{2}) & Z_o p\ C_T[D(p) + \frac{2k^2}{\gamma p}\tanh(\frac{\gamma p}{2})] \end{bmatrix} \quad (5.5)$$

where

$$C_T = N\ C_o, \quad (5.6)$$

$$D(p) = 1 - \frac{2K^2}{\gamma p} \tanh\ (\frac{\gamma p}{2}), \quad (5.7)$$

$$\sinh\ (\delta(p)) = \frac{[\ \{\ \sinh \gamma p + \frac{2K^2}{\gamma p}(1 - \cosh \gamma p)\}\ \sinh \gamma p]^{\frac{1}{2}}}{(1 - \frac{K^2}{\gamma p} \sinh \gamma p)}, \quad (5.8)$$

and

$$K^2 = \frac{\alpha Y_o \gamma}{C_s} = \alpha k^2 \quad (5.9)$$

6. TRANSFER SCATTERING MATRIX FOR NON-UNIFORM AND UNIFORM SAW TRANSDUCERS

An important description of interest and often helpful for design purposes of various SAW devices is the transfer scattering matrix for the transducer. The transfer scattering matrix T, defined as in [12], can be expressed in terms of the scattering matrix elements S_{ij}'s of the S matrix as shown.

$$T = \begin{bmatrix} S_{12} - S_{11}S_{21}^{-1}S_{22} & S_{11}S_{21}^{-1} \\ -S_{21}^{-1}S_{22} & S_{21}^{-1} \end{bmatrix} \quad (6.1)$$

We can easily calculate the transfer scattering matrix using the Equation (6.1) for non-uniform and uniform SAW transducers expressed in the hybrid, crossed-field and in-line model forms.

Based on the admittance matrix elements of a non-uniform transducer, as given by the Equations (3.30) through (3.38), we can directly obtain the scattering and hence the transfer scattering matrices for the same device. However, it is obvious that the resulting expressions will have forms which are very complex in nature, and thus the use of digital computer will be appropriate in calculating the transfer scattering matrix elements for non-uniform SAW transducers. Since in most practical situations it is convenient to deal with the transducers consisting of identical basic sections, we will explicitly derive the expressions for the transfer scattering matrix for uniform transducers.

For a transducer consisting of N equal basic sections, $S_{11} = S_{22}$ and $S_{12} = S_{21}$. Therefore the transfer scattering matrix for a uniform transducer takes the form

$$T = S_{12}^{-1} \begin{bmatrix} (S_{12}^2 - S_{11}^2) & S_{11} \\ -S_{11} & 1 \end{bmatrix} \tag{6.2}$$

The use of the Equation (5.5) in (4.12) to (4.17) gives the scattering matrix shown elements as given below.

$$S_{11} = S_{22} = \frac{1}{M} [(1 - \frac{Y_o^2}{D}) \{1 + pC_T[1 + \frac{2k^2}{\gamma pD} \tanh (\frac{\gamma p}{2})]\}$$

$$+ \frac{2\phi^2 Y_o}{D^{5/2}} \tanh^2 (\frac{\gamma p}{2}) \{\text{ctnh} (N\delta) + (-1)^{N+1} \text{csch}(n\delta)\}] \tag{6.3}$$

$$S_{12} = S_{21} = \frac{2}{M} [\frac{Y_o}{D^{1/2}} \text{csch} (N\delta) \{1 + pC_T[1 + \frac{2k^2}{\gamma pD} \tanh (\frac{\gamma p}{2})]\}$$

$$+ (-1)^{N+1} \frac{\phi^2 Y_o^2}{D^2} \tanh^2 (\frac{\gamma p}{2})] \tag{6.4}$$

$$S_{13} = S_{31} = - \frac{2\ Y_o}{M} [\frac{\phi}{D} \tanh (\frac{\gamma p}{2}) + \frac{\phi\ Y_o}{D^{3/2}} \tanh (\frac{\gamma p}{2})$$

$$\cdot \{\text{ctnh} (N\delta) + (-1)^{N+1} \text{csch}(N\delta)\}] \tag{6.5}$$

$$S_{23} = S_{32} = (-1)^{N-1} S_{13} \tag{6.6}$$

$$S_{33} = \frac{1}{M} [\{1 - pC_T[1 + \frac{2k^2}{\gamma pD} \tanh (\frac{\gamma p}{2})]\} \{(1 + \frac{Yo^2}{D}) + \frac{2Y_o}{D^{1/2}} \text{ctnh} (N\delta)\}$$

$$+ \frac{2\phi^2 Y_o^2}{D^2} \tanh^2 (\frac{\gamma p}{2}) + \frac{2\phi^2 Y_o^3}{D^{5/2}} \tanh^2 (\frac{\gamma p}{2}) \{\text{ctnh} (N\delta) + (-1)^{N+1} \text{csch} (N\delta)\}] \tag{6.7}$$

where

$$M = [\{1 + pC_T[1 + \frac{2k^2}{\gamma pD} \tanh (\frac{\gamma p}{2})]\} \{(1 + \frac{Y_o^2}{D}) + \frac{2Y_o}{D^{1/2}} \text{ctnh} (N\delta)\}$$

$$- \frac{2\phi^2 Y_o^2}{D^2} \tanh^2 (\frac{\gamma p}{2}) - \frac{2\phi^2 Y_o^3}{D^{5/2}} \tanh^2 (\frac{\gamma p}{2}) \{\text{ctnh} (N\delta) + (-1)^{N+1} \text{csch} (N\delta)\}] \tag{6.8}$$

and D is given by the Equation (5.7).

6.1. N-SECTION HYBRID MODEL TRANSDUCERS

By the use of Equations (6.3) and (6.4), the transfer scattering matrix elements T_{ij}'s for a uniform transducer consisting of N sections of the hybrid model are obtained in the following form.

$$T_{11} = \frac{1}{M\Delta_h} [(1 + pC_T\, Q_h(p))^2 \{\frac{4Y_o^2}{D} \operatorname{csch}^2 (N\delta) - (1 - \frac{Y_o^2}{D})^2\}$$

$$+ 4 (\frac{\phi Y_o}{D}) \tanh (\frac{\gamma p}{2}))^4 \quad \{1 - \frac{Y_o^2}{D} [\operatorname{ctnh} (N\delta) + (-1)^{N+1} \operatorname{csch} (N\delta)]^2\}$$

$$+ \frac{4Y_o}{D^{1/2}} (\frac{\phi Y_o}{D} \tanh (\frac{\gamma p}{2}))^2 (1 + pC_T\, Q_h(p))$$

$$\times\{2(-1)^{N+1} \operatorname{csch} (N\delta) - (1 - \frac{Y_o^2}{D}) [\operatorname{ctnh} (N\delta) + (-1)^{N+1} \operatorname{csch} (N\delta)]\}] \tag{6.9}$$

$$T_{12} = \frac{1}{\Delta_h} [(1 - \frac{Y_o^2}{D}) (1 + pC_T Q_h(p)) + \frac{2Y_o}{D^{1/2}} (\frac{\phi Y_o}{D} \tanh (\frac{\gamma p}{2}))^2$$

$$\cdot\{\operatorname{ctnh} (N\delta) + (-1)^{N+1} \operatorname{csch} (N\delta)\}] \tag{6.10}$$

$$T_{12} = -T_{12} \tag{6.11}$$

$$T_{22} = M/\Delta_h \tag{6.12}$$

where

$$Q_h(p) = 1 + \frac{2k^2}{\gamma p D(p)} \tanh (\frac{\gamma p}{2}) \tag{6.13}$$

$$\Delta_h = [\frac{2Y_o}{D^{1/2}} \operatorname{csch} (N\delta) \{1 + pC_T Q_h(p)\} + 2(-1)^{N+1} (\frac{\phi Y_o}{D} \tanh (\frac{\gamma p}{2}))^2] \tag{6.14}$$

and M is given by the Equation (6.8).

It is also clear that the results given by the Equations (6.9) through (6.14) are valid for both the crossed-field model, $\alpha = 0$, of a basic section. For the in-line model there is essentially no simplification in the results except $K^2 = \alpha k^2$ is just replaced by k^2. However in the crossed-field model situation considerable simplification results.

6.2 N-SECTION CROSSED-FIELD MODEL TRANSDUCERS

Setting $\alpha = 0$ for the crossed-field case gives

$$K = 0, \quad D(p) = 1 \tag{6.15}$$

and hence, by Equation (5.8),

$$\delta(p) = \gamma p. \tag{6.16}$$

By specializing the Equations (6.9) to (6.14) we have the transfer scattering matrix for transducers consisting of N basic sections of the crossed-field model as shown.

$$T_{11} = \frac{1}{M\,\Delta_c}\,[(1 + pC_TQ_c(p))^2\{4Y_o^2\ \mathrm{csch}^2(N\gamma p) - (1-Y_o^2)\}$$

$$+\ 4\phi^2Y_o^4\ \tanh^4(\tfrac{\gamma p}{2})\ .\ \{1 - Y_o^2[\mathrm{ctnh}\ (N\gamma p) + (-1)^{N+1}\mathrm{csch}\ (N\gamma p)]^2\}$$

$$+\ 4\phi^2Y_o^3\ \tanh^2(\tfrac{\gamma p}{2})\ \{1 + pC_TQ_c(p)\}\ .\ \{2(-1)^{N+1}\mathrm{csch}\ (N\gamma p)$$

$$-\ (1-Y_o^2)\ [\mathrm{ctnh}\ (N\gamma p) + (-1)^{N+1}\mathrm{csch}\ (N\gamma p)]\}] \tag{6.17}$$

$$T_{12} = \frac{1}{\Delta_c}\ [\ (1 - Y_o^2)\ \{1 + pC_TQ_c(p)\} + 2\phi^2Y_o^3\ \tanh^2(\tfrac{\gamma p}{2})$$

$$\cdot\{\mathrm{ctnh}\ (N\gamma p) + (-1)^{N+1}\mathrm{csch}\ (N\gamma p)\}] \tag{6.18}$$

$$T_{21} = -T_{12} \tag{6.19}$$

$$T_{22} = M/\Delta_c \tag{6.20}$$

with

$$Q_c(p) = 1 + \frac{2k^2}{\gamma p}\tanh\ (\tfrac{\gamma p}{2}) \tag{6.21}$$

$$\Delta_c = [2Y_o\ \mathrm{csch}\ (N\gamma p)\ \{1 + pC_TQ_c(p)\} + 2(-1)^{N+1}(\phi Y_o\tanh\ (\tfrac{\gamma p}{2}))^2] \tag{6.22}$$

and

$$M = [\{1 + pC_T + 2N\phi^2Y_o\ \tanh\ (\tfrac{\gamma p}{2})\}\ \{1 + Y_o^2 + 2Y_o\ \mathrm{ctnh}\ (N\gamma p)\}]$$

$$-\ 2(\phi Y_o\tanh\ (\tfrac{\gamma p}{2}))^2[1 + Y_o\{\mathrm{ctnh}\ (N\gamma p) + (-1)^{N+1}\mathrm{csch}\ (N\gamma p)\}] \tag{6.23}$$

7. CONCLUSIONS

The explicit expressions for the transfer scattering matrix of a SAW interdigital transducer consisting of N nonuniform basic sections can be obtained with the help of the general expressions derived in this paper. The product matrix Φ_R in the nonuniform case is not easy to calculate in a straightforward analytical way, and becomes almost impossible for a large number of sections without the use of a digital computer. However, the general forms presented for the matrix elements are suitable for computer aided deisgn of SAW transducers, and therefore the existence of the present analysis will be very helpful for more extensive design formalism of surface acoustic wave devices.

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